Assessing the Model Risk with Respect to the Interest Rate Term Structure under Solvency II

Michael Martin

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ABSTRACT

Interest rate risk, i.e. the risk of changes in the interest rate term structure, is of high relevance in insurers' risk management. Due to large capital investments in interest rate sensitive assets such as bonds, interest rate risk plays a considerable role for deriving the solvency capital requirement (SCR) in the context of Solvency II. In addition to the Solvency II standard model, we apply the model of Gatzert and Martin (2012) for introducing a partial internal model for the market risk of bond exposures. The aim of this paper is to assess model risk with focus on bonds in the market risk module of Solvency II regarding the underlying interest rate process and input parameters. After introducing calibration methods for short rate models, we quantify interest rate and credit risk for corporate and government bonds and demonstrate that the type of process can have a considerable impact despite comparable underlying input data. The results show that, in general, the SCR for interest rate risk derived from the standard model of Solvency II tends to the SCR achieved by the short rate model from Vasicek (1977), while the application of the Cox, Ingersoll, and Ross (1985) model leads to a lower SCR. For low-rated bonds, the internal models approximate each other and, moreover, show a considerable underestimation of credit risk in the Solvency II model.

[*] Michael Martin is at the Friedrich-Alexander-University (FAU) of Erlangen-Nürnberg, Chair for Insurance Economics, Lange Gasse 20, 90403 Nürnberg, Germany. Email: michael.martin@wiso.uni-erlangen.de.

1. INTRODUCTION

Due to the large capital investments in interest rate sensitive assets in the insurance industry, the measurement of market risks, arising from changes in the interest rate term structure, is of great importance. The European supervisory system Solvency II provides one separate sub-module as a part of the market risk module for quantifying interest rate risk in the latest proposed standard model of 2010/2011. In 2011, the Federal Financial Supervisory Authority in Germany (BaFin) published the results of the quantitative impact study QIS 5, applying the standard model of 2010/2011, for the German insurance industry. The reporting reflects the high relevance of interest rate risk in particular for life and health insurers (see BaFin, 2011, pp. 16, 18, 21).[1] Therefore, the aim of this is paper is to quantify the risk for corporate and government bond investments with a partial internal risk model that accounts for interest rate and credit risk and thereby focusing on different interest rate processes for model risk. We concentrate on the asset class of corporate and government bonds, where the credit risk (including spread risk) has to be quantified beside interest rate risk. Furthermore, diversification benefits by integrating both, the interest rate and credit risk, in an internal model are analyzed and the sensitivity of misestimation by calibrating the input parameters is identified.

The European Union (EU) solvency regulation Solvency II, characterized by a three pillar structure in accordance with the European banking supervision Basel II/III, will impose risk-based capital requirements for insurance companies. In general, besides a scenario-based standard model, the regulators allow for deriving solvency capital requirements (SCR) with a fully internal or partial internal approach (see European Parliament and the Council, 2009, Article 112, No. 1 to 7). The different methods deriving the SCR are described in detail in Gatzert and Wesker (2011). The standard model of Solvency II is structured by a bottom-up approach consisting of six modules to calculate the basic SCR (BSCR). Gatzert and Martin (2012) examine the market risk module, covering the risk resulting from the variation of the market value of financial instruments, and therein the sub-modules equity risk, interest rate risk and credit risk. They propose a partial internal model that fully accounts for equity, interest rate and credit risk for the asset class of stocks and bonds. Consequences for the insurers’

[1] Following the results of QIS 5 from the BaFin for the German insurance industry, the SCR of the market risk module is identified as the key risk in the life and health insurance sector with 82% and 81% respectively of the BSCR. Within the market risk module of the Solvency II standard approach, the interest rate risk dominates with 64% (life) and 51% (health), followed by the spread risk sub-module with 31% (life) and 45% (health). In the property-casualty sector, the market risk module takes the second largest position for the BSCR (48%) where interest rate risk (20%) as well as spread risk (29%) are also of relevance (see BaFin, 2011, pp. 16, 18, 21). All values without diversification.

investment behavior, caused by the new European solvency regulations, are analyzed by Fitch Ratings (Piozot et al., 2011). First, one main result is an increasing attractiveness and thus an increasing investment in higher-rated corporate and government bonds in general. Especially bond exposures with shorter maturities will be preferred due to the duration-based approach in Solvency II (see also Gatzert and Martin, 2012).[2] Second, Fitch Ratings (Piozot et al., 2011) prognosticate a rising demand for government bonds issued by countries of the European Economic Area (EEA) or borrowings guaranteed by one of these states caused by a special treatment for these assets. Following Solvency II, no solvency capital for credit risk is required for investments from bond issuers that are members of the EEA as mentioned before, e.g. Greece or Ireland (see CEIOPS, 2010).

In the literature, stochastic interest rate models are implemented, e.g. in Gerstner et al. (2008) and Graf, Kling, and Russ (2011) to specify the insurer's asset portfolio process of interest sensitive investments. While Gerstner et al. (2008) introduce the model from Cox, Ingersoll, and Ross (1985) in a life insurance asset-liability model, the short rate model from Vasicek (1977) is applied by Graf, Kling, and Russ (2011) regarding the asset dynamic in the context of participating life insurance. In general, the literature on short term interest rate models and the valuation of interest derivates, such as bonds, is rather extensive (see, e.g. Merton, 1973, Vasicek, 1977, Brennan and Schwartz, 1977, Dothan, 1978, Cox, Ingersoll, and Ross, 1980, Brennan and Schwartz, 1979, Cox, Ingersoll, and Ross, 1985, Ho and Lee, 1986, Hull and White, 1990, and Pearson and Sun, 1994). Furthermore, several papers have focused on the comparison and estimation of different short rate dynamics. Chan et al. (1992) empirically investigate different models for the short rate of interest fitted on the same data in one econometric framework. This procedure allows them to identify and compare the performance of eight interest rate models. Further investigations of short rate models and the calibration can also be found, e.g. in Nowman (1998), de Jong and Santa-Clara (1999), Duan and Simonato (1999), and Faff and Gray (2006). Song et al. (2012) examine the behavior of various short rate models concerning bond investments. In their analysis, they exclude credit risk but focus on risk measurement as the Value at risk and the expected shortfall in terms of interest rate risk. To account for credit risk of defaultable bond exposures, reduced-form credit risk models are extensively researched in the literature (see, e.g. Jarrow and Turnbull, 1995, Das and Tufano, 1996, Jarrow, Lando, and Turnbull, 1997, Duffie and Singleton, 1997, Lando, 1998, and Duffie and Singleton, 1999).

[2] Gatzert and Martin (2012) also introduce model risk but limit the analysis to single bonds with 5 years maturity applying Cox, Ingersoll, and Ross (1985) and Vasicek (1977).

In this paper, we focus on the asset side of an insurer and quantify the solvency capital requirements for corporate and government bond investments in terms of interest rate and credit risk. The solvency capital requirement is calculated with a partial internal model as well as the standard model of Solvency II (QIS 5). In this context, particular attention is paid to model risk of the SCR for interest rate risk, where the changes in the term structure of interest are determined by a short term interest rate process. Thus, the question arises which model risk results from the model selection of the short rate dynamic in the partial internal approach with respect to the solvency capital requirement for bond investments, also compared to the Solvency II standard model in the current calibration (QIS 5). For quantifying the market risk of bond exposures, two types of risk, interest rate and credit risk, have to be distinguished. Based on the partial internal model for the SCR of bonds introduced by Gatzert and Martin (2012), we extent their work by focusing and comparing the application of different one factor short term interest rate models. Beside the Solvency II standard approach, three interest rate models in a continuous time framework are considered to derive the risk-free interest term structure. Besides the models' characteristics, calibration methods are introduced. In the numerical analysis, we concentrate on the interest rate models from Merton (1973), Vasicek (1977) and Cox, Ingersoll, and Ross (1985), which are preferred in the bond valuation literature due to the affine term structure to calculate bond prices analytically. In addition, to integrate credit risk, the rating-based credit risk model published by Jarrow, Lando, and Turnbull (1997) is adopted. The model of Jarrow, Lando, and Turnbull (1997) describes the rating transition or rather default process of defaultable exposures with respect to credit quality. This process is defined by a time-homogenous Markov chain with a distribution determined by transition rates. Finally, the dimension of diversification benefit between interest rate and credit risk is analyzed.

Our findings show that model selection of the short rate process can lead to a considerable change in the solvency capital requirements for interest rate risk with a partial internal model for corporate and government bonds due to particular characteristics of the different interest rate models. The influence of interest rate risk to the total capital requirements for the market risk of bonds is strongly dependent on the credit rating of the considered bonds. For bonds with higher credit quality and higher maturities, the numerical analysis shows that the SCR differ substantially when changing the underlying short rate model. Speculative grade credit quality assets are primarily affected by credit risk so that the interest rate model choice is nearly independent from the solvency capital requirements.

The remainder of the paper is structured as follows. The short rate modeling framework, the properties of affine term structure models and methods for calibration are introduced in Section 2. The procedure for the SCR quantification of bond exposures with an internal model approach and the standard model of Solvency II are presented in Section 3. Section 4 contains the results of the numerical analysis, and Section 5 concludes.

2. SHORT RATE MODELS: A TIME-HOMOGENOUS FRAMEWORK

2.1 One factor short rate models in continuous time

For modeling the interest rate term structure in continuous time, we consider one factor short rate models, where the short rate of interest is the only explanatory variable. The diffusion process for the risk-free short rate $r(t)$ is an Itô process defined by the stochastic differential equation

$$dr(t) = \hat{\mu}_r\left(t, r(t)\right)dt + \hat{\sigma}_r\left(t, r(t)\right)dW_r^{\mathbb{P}}(t) \tag{1}$$

with drift function $\hat{\mu}_r\left(t, r(t)\right)$, diffusion function $\hat{\sigma}_r\left(t, r(t)\right)$, and $W_r^{\mathbb{P}}$ being a standard Brownian motion on the probability space $\left(\Omega_r, \mathcal{F}_r, \mathbb{P}\right)$ with filtration $\mathcal{F}_r$ and real world probability measure $\mathbb{P}$ (see Brigo and Mercurio, 2007).

The price of a non-defaultable zero coupon bond $p(h,t)$ at time t that pays out one monetary unit at time h, $t \leq h$ and $p(h,h)=1$, is defined by the short rate $r(t)$. Under the real world measure $\mathbb{P}$ with the short rate dynamic as given in Equation (1), the price process is exhibited by the stochastic differential equation

$$dp(t,h) = p(t,h)\cdot r(t)dW^{\mathbb{P}} = p(t,h)\cdot\left(\hat{\mu}_p\left(t, r(t)\right)dt + \hat{\sigma}_p\left(t, r(t)\right)dW^{\mathbb{P}}\right). \tag{2}$$

The drift $\hat{\mu}_r\left(t, r(t)\right)$ and the diffusion $\hat{\sigma}_r\left(t, r(t)\right)$ of the bond price process exhibited in Equation (2) are obtained by using Itô's theorem:[3]

$$\hat{\mu}_p\left(t, r(t)\right) = \frac{1}{p}\left(\frac{\partial p}{\partial t} + \hat{\mu}_r\frac{\partial p}{\partial r} + \frac{1}{2}\cdot\hat{\sigma}_r^2\frac{\partial^2 p}{\partial r^2}\right) \text{ and } \hat{\sigma}_p\left(t, r(t)\right) = \frac{1}{p}\cdot\hat{\sigma}_r\frac{\partial p}{\partial r}. \tag{3}$$

3 Here, we denote $p = p(t,h)$, $r = r(t)$, $\hat{\mu}_r = \hat{\mu}_r\left(t, r(t)\right)$ and $\hat{\sigma}_r^2 = \hat{\sigma}_r^2\left(t, r(t)\right)$.

Under the assumption of an arbitrage-free bond market, the change from the real world measure to the risk-neutral measure is determined by the market price of risk $\lambda(t)$. The market price of risk represents a risk premium and is given by (see Vasicek, 1977)

$$\lambda\left(t, r(t)\right) = \frac{\hat{\mu}_p\left(t, r(t)\right) - r(t)}{\hat{\sigma}_p\left(t, r(t)\right)}. \tag{4}$$

Thus, the $\mathbb{Q}$-dynamic of the one factor diffusion model in Equation (1) can be exhibited by the process

$$dr(t) = \mu_r\left(t, r(t)\right)dt + \sigma_r\left(t, r(t)\right)dW_r^{\mathbb{Q}}(t) \tag{5}$$

where $W_r^{\mathbb{Q}}$ denotes a standard Brownian motion under the risk-neutral probability measure $\mathbb{Q}$ (see Cairns, 2004) with

$$\mu_r\left(t, r(t)\right) = \hat{\mu}_r\left(t, r(t)\right) - \lambda(t) \cdot \hat{\sigma}_r\left(t, r(t)\right) \text{ and } \sigma_r\left(t, r(t)\right) = \hat{\sigma}_r\left(t, r(t)\right)$$

as a consequence of the condition in Equation (4) and the Girsanov theorem[4] (see Bingham and Kiesel, 2010). The price process of a zero coupon bond in an arbitrage-free world under $\mathbb{Q}$ is given by

$$p(t,h) = E_t^{\mathbb{Q}}\left(e^{-\int_t^h r(s)ds}\right) \tag{6}$$

on the probability space $\left(\Omega_r, \mathcal{F}_r, \mathbb{Q}\right)$ with the risk-neutral dynamic for the spot rate $r(t)$ from Equation (1) (see Brigo and Mercurio, 2007). When we equate the drift function of the bond price process $\hat{\mu}_p\left(t, r(t)\right)$ in Equation (3) and (4), the fundamental differential equation for interest rate derivatives in an arbitrage-free world is obtained. The so called “term structure equation”,

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mu_r \cdot \dfrac{\partial p}{\partial r} + \dfrac{1}{2} \cdot \sigma_r^2 \dfrac{\partial^2 p}{\partial r^2} - p \cdot r = 0 \\ p(h,h) = 1 \end{cases}, \tag{7}$$

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4 See Girsanov (1960).

solvable by using the Feynman-Kac formula[5] with boundary condition $p(h,h)=1$ (see Cairns, 2004). The class of affine term structure interest rate models has the property to solve the term structure equation (see Equation (7)) in an arbitrage-free market. The bond price can be represented by an equation of the from

$$p(t,h) = e^{A(t,h)-B(t,h)\cdot r(t)}, \tag{8}$$

where $A(t,h)$ and $B(t,h)$ are functions of time t and end time h. The process in Equation (8) admits an affine term structure model, if $A(t,h)$ and $B(t,h)$ satisfy the system (see Björk, 2009)

$$\begin{cases} \dfrac{\partial A}{\partial t} = \dfrac{1}{2}\cdot c(t)\cdot B^2 - a(t)\cdot B \\ A(h,h) = 0 \end{cases} \tag{9}$$

and

$$\begin{cases} \dfrac{\partial B}{\partial t} = \dfrac{1}{2}\cdot d(t)\cdot B^2 - b(t)\cdot B - 1 \\ B(h,h) = 0, \end{cases} \tag{10}$$

given by the term structure equation in Equation (7). Setting $t = h$, the affine representation of the zero coupon bond price in Equation (8) satisfy the second condition in Equation (9) and (10) with $A(t,h) = B(t,h) = 0$. Given a short rate dynamic as in Equation (5), a sufficient condition for the existence of an affine term structure is that the drift and volatility of the short rate process can be formed to (see Cairns, 2004)

$$\mu_r(t,r(t)) = \alpha(t) + \beta(t)\cdot r(t) \text{ and } \sigma_r(t,r(t)) = \sqrt{\gamma(t)\cdot r(t) + \delta(t)}. \tag{11}$$

The functional form of $\mu_r(t,r(t))$ and $\sigma_r(t,r(t))$ in Equation (11) results in different one factor short rate models for modeling the interest tern structure. Regarding time-invariant short rate models, the dynamic depends only on constant coefficients. Thus the parameters of the process can be written as[6]

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[5] See, e.g. McNeil, Frey, and Embrechts (2005).

[6] The assumption of time-homogenous short rate models also implicate $\alpha(t)=\alpha$, $\beta(t)=\beta$, $\gamma(t)=\gamma$, and $\delta(t)=\delta$ in Equation (11).

$\mu_r(t, r(t)) = \mu_r(r(t))$ and $\sigma_r(t, r(t)) = \sigma_r(r(t))$.

Time-homogenous short rate models are Markov and have an analytical solution for bond prices (see Equation (8)) when they offer an affine term structure.

2.2 Time-homogenous one factor short term interest rate models and characteristics

In the literature there exist several approaches for modeling the interest rate term structure depending on one factor in a time-homogenous framework. In general, the models differ in the specification of the drift function $\mu_r(r(t))$ and the diffusion function $\sigma_r(r(t))$, implying different process features. Table 1 summarizes the most familiar time-homogenous one factor short term interest rate models in the literature. Beside the drift function $\mu_r(r(t))$ and the diffusion function $\sigma_r(r(t))$ of the stochastic process, Table 1 gives information about further key characteristics of each process. The expression $r(t) \geq 0$ indicates whether the process always produces positive short rates (Y) or not (N). The allowance for negative interest rates is not tenable in practice, so that the characteristic of positive interest rates is a desired property. Furthermore, we consider the existence of an affine term structure for bond prices (ATS) for an analytical solution. The existence of an analytical bond price solution brings major advantages with respect to computational calculations.

As a result of the computational features, the short term interest rate models with an affine term structure to calculate bond prices analytically, i.e. Merton (1973), Vasicek (1977) and Cox, Ingersoll, and Ross (1985), are the most famous approaches in the bond price literature. In particular, the models from Vasicek (1977) and Cox, Ingersoll, and Ross (1985), both offering a mean reverting drift function, play a significant role in the literature. Merton (1973) develop an affine model for the short term interest rate with a constant drift function $\mu_r(r(t))$ and a constant diffusion function $\sigma_r(r(t))$. As a consequence of this setting, the process' first two moments are boundless and, thus, allow for negative interest rates (see Table 1). Another approach is given by Vasicek (1977) by introducing the desirable and realistic property of mean reverting instead of a constant drift function. The diffusion function is assumed to be constant, just as the model of Merton (1973). Hence, the first two moments of the process converge to a long-term mean respectively long-term variance. However, a major drawback of the model from Vasicek (1977) is also the allowance of negative interest rates with a positive probability. Cox, Ingersoll, and Ross (1985) propose a further affine term structure model where the drift function as well as the volatility function is defined proportional to the short rate instead of a constant setting. The consequence is the providing of strict positive interest rates (under one additional condition) and the property of mean reversion in the short rate model from Cox, Ingersoll, and Ross (1985).

Table 1: One factor short rate models in a time-homogenous framework (selection)

	$\mu_r(r(t))$	$\sigma_r(r(t))$	$r(t) \geq 0$	ATS
Merton (1973)	θ	σ	N	Y
Vasicek (1977)	$\kappa(\theta - r(t))$	σ	N	Y
Dothan (1978)[7]		$\sigma \cdot r(t)$	Y	N
Cox, Ingersoll, and Ross (1980)		$\sigma \cdot r(t)^{\frac{3}{2}}$	Y	N
Brennan and Schwartz (1979)[8]	$\kappa(\theta - r(t))$	$\sigma \cdot r(t)$	Y	N
Rendleman and Bartter (1980)[9]	$\theta \cdot r(t)$	$\sigma \cdot r(t)$	Y	N
Cox, Ingersoll, and Ross (1985)	$\kappa(\theta - r(t))$	$\sigma \cdot \sqrt{r(t)}$	Y	Y
Pearson and Sun (1994)	$\kappa(\theta - r(t))$	$\sigma \cdot \sqrt{r(t) - \beta}$	Y*	N

** The short rate model from Pearson and Sun (1994) generates always positive values if $\beta > 0$.*
Notes: ATS: affine term structure for bond prices; Y: yes; N: no.

2.3 Time-homogenous one factor short rate models with affine term structure

Merton (1973)

The short rate dynamic consisting of a Brownian motion with a constant drift function is introduced by Merton (1973). This process is defined by setting the parameter in Equation (11) to

$$\alpha = \theta,\ \ \beta = \gamma = 0 \ \text{ and } \ \delta = \sigma^2$$

with strict positive σ^2. Thus, the stochastic differential equation for the short term interest rate in an arbitrage-free market is given by

$$dr(t) = \theta dt + \sigma dW_r^{\mathbb{Q}}(t).$$

[7] Dothan (1978) proposes a short rate process for discount bond pricing. In the work of Brennan and Schwartz (1977), savings bonds, retractable bonds and callable bonds are priced.

[8] The model of Dothan (1978) is extended by Brennan and Schwartz (1979) for pricing discount bonds. Courtadon (1982) use the model for convertible bond price valuation.

[9] In the literature, the model from Rendleman and Bartter (1980), also addressed by Marsh and Rosenfeld (1983), is also assigned to Dothan (1978).

The short term interest rate model from Merton (1973) implies the assumption of a normal distributed process with short rate $r(t)$ at time t with information given at time s, $s \leq t$, exhibited by

$$r(t)\big|\mathcal{F}_s \sim \mathcal{N}\left(r(s)+\theta\cdot(t-s),\sigma^2\cdot(t-s)\right),$$

where $\left(\mathcal{F}_{r,t}\right)_{t\in\mathbb{N}_0}$ denotes the filtration generated by the Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. As a result of the infinity of the first two moments of the distribution,

$$\lim_{t\to\infty} E\left(r(t)\big|\mathcal{F}_s\right)=\pm\infty \text{ and } \lim_{t\to\infty} Var\left(r(t)\big|\mathcal{F}_s\right)=\infty,$$

the process allows negative and infinite interest rates. The stochastic differential equation for affine term structure models in Equation (9) and (10) can be solved by

$$A(t,h)=\frac{1}{6}\cdot\sigma_r^2\cdot(h-t)^3-\frac{1}{2}\cdot\theta\cdot(h-t)^2 \text{ and } B(t,h)=h-t$$

with boundary condition $A(t,h) = B(t,h) = 0$ (see Merton, 1973). Additionally, the model of Merton (1973) assumes a constant market price of risk with $\lambda\left(t,r(t)\right)=\lambda_0$. Hence, the short rate dynamics for this model under the real world probability measure $\mathbb{P}$ follows

$$\begin{aligned} dr(t) &= \left(\theta+\lambda_0\cdot\sigma\right)dt+\sigma dW_r^{\mathbb{P}}(t) \\ &= \hat{\theta}dt+\sigma dW_r^{\mathbb{P}}(t) \end{aligned}$$

with the condition $\theta=\hat{\theta}-\lambda_0\cdot\sigma$.

Vasicek (1977)

Vasicek (1977) proposes a model for the interest rate as an Ornstein-Uhlenbeck process where the drift function of the process is characterized by the property of mean reverting. When the short rate moves under or above the long-term mean, the process reverts to the mean with a given speed of mean revision. The short rate dynamic from Vasicek (1977) is exhibited by

$$dr(t)=\kappa\cdot\left(\theta-r(t)\right)dt+\sigma dW_r^{\mathbb{Q}}(t)$$

under risk-neutral probability measure $\mathbb{Q}$ and with the parameters derived from Equation (11),

$\alpha = \kappa \cdot \theta$, $\beta = -\kappa$, $\gamma = 0$ and $\delta = \sigma^2$,

where κ; σ, and θ are strictly positive. Here, κ controls the speed of the mean reversion to the long-term mean θ (see Cairns, 2004). The volatility of the short term interest rate process is represented by σ and the (conditional) short rate $r(t)$ in this model framework is normal distributed with

$$r(t)\big|\mathcal{F}_s \sim \mathcal{N}\left(r(s)\cdot e^{-\kappa\cdot(t-s)} + \theta\cdot\left(1-e^{-\kappa\cdot(t-s)}\right), \frac{\sigma^2}{2\cdot\kappa}\cdot\left(1-e^{-2\cdot\kappa\cdot(t-s)}\right)\right)$$

and the filtration $\left(\mathcal{F}_{r,t}\right)_{t\in\mathbb{N}_0}$, generated by the Brownian motion under $\mathbb{Q}$. Concerning the mean and variance of the short rate process for $t\to\infty$ from the Vasicek (1977) model,

$$\lim_{t\to\infty} E\left(r(t)\big|\mathcal{F}_s\right) = \theta \text{ and } \lim_{t\to\infty} Var\left(r(t)\big|\mathcal{F}_s\right) = \frac{\sigma^2}{2\cdot\kappa},$$

the mean for $t\to\infty$ just corresponds to the long-term mean θ with long-term variance $\sigma^2/(2\cdot\kappa)$. As a consequence, the interest rates can become negative with positive probability in this process (see Brigo and Mercurio, 2007). The Vasicek (1977) model has a solution for the stochastic differential equation for affine term structure models in Equation (9) and (10) by

$$A(t,h) = \left(B(t,h)-(h-t)\right)\cdot\left(\theta - \frac{\sigma^2}{2\cdot\kappa^2}\right) - \frac{\sigma^2}{4\cdot\kappa}\cdot B(t,h)^2 \text{ and } B(t,h) = \frac{1-e^{-\kappa\cdot(h-t)}}{\kappa}$$

with the boundary condition $A(t,h) = B(t,h) = 0$ (see Björk, 2009). Under the real world probability measure and a constant market price of risk $\lambda\left(t, r(t)\right) = \lambda_0$, the short rate dynamic changes to (see Vasicek, 1977)

$$\begin{aligned} dr(t) &= \left(\kappa\cdot\left(\theta - r(t)\right) - \lambda_0\cdot\sigma\right)dt + \sigma dW_r^{\mathbb{P}}(t) \\ &= \hat{\kappa}\cdot\left(\hat{\theta} - r(t)\right)dt + \sigma dW_r^{\mathbb{P}}(t). \end{aligned}$$

Thus, the relationship between the empirical and the risk-neutral parameter is given by $\kappa = \hat{\kappa}$ and $\theta = \hat{\theta} - (\lambda_0 \cdot \sigma)/\kappa$.[10]

Cox, Ingersoll, and Ross (1985)

A further model for the short term interest rate in a time-homogenous framework is developed by Cox, Ingersoll, and Ross (1985), known as “square root process”. They adapt the drift function similar to the Ornstein-Uhlenbeck process. However, to avoid negative interest rates as in the case of Vasicek (1977), Cox, Ingersoll, and Ross (1985) assume a time-varying volatility function. The root function of the short rate is integrated in the volatility, so the variance increases (decreases), if the short term increases (decreases). Accordingly, the parameter specification for the sufficient condition for the existence of an affine term structure (see Equation (11)) is given by

$$\alpha = \kappa \cdot \theta,\ \beta = -\kappa,\ \gamma = \sigma^2 \text{ and } \delta = 0.$$

The short rate process can be described by

$$dr(t) = \kappa \cdot (\theta - r(t))\,dt + \sigma \cdot \sqrt{r(t)}\,dW_r^{\mathbb{Q}}(t)$$

with speed of mean reversion κ, long-term mean θ and volatility σ, all strict positive, and a standard $\mathbb{Q}$-Brownian motion $W_r^{\mathbb{Q}}$. Following Cox, Ingersoll, and Ross (1985), the process provides a strictly positive short rate for all *t*, if the condition $2 \cdot \kappa \cdot \theta \geq \sigma^2$ is satisfied. This is proved by considering the first and second moment of the short rate *r*(*t*) at time *t*, with given information at time *s*, $s \leq t$:

$$\lim_{t \to \infty} E\left(r(t) \middle| \mathcal{F}_s\right) = \theta \text{ and } \lim_{t \to \infty} Var\left(r(t) \middle| \mathcal{F}_s\right) = \theta \cdot \left(\frac{\sigma^2}{2 \cdot \kappa}\right).$$

The conditional probability density of the short rate process *r*(*t*) is given by an non-central chi-squared distribution $\chi^2(2 \cdot c \cdot r(s); 2 \cdot q + 2, 2 \cdot u)$ with $2 \cdot q + 2$ degrees of freedom and non-central parameter $2 \cdot u$ (see Cox, Ingersoll, and Ross, 1985).[11] Furthermore, the short rate dynamic satisfy the (necessary) condition for an affine term structure representation (see Equation (9) and (10)) is exhibited by

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10 Alternately, the parameter can be set to $\kappa = \hat{\kappa} - \lambda_0 \cdot \sigma^2$ and $\theta = \hat{\kappa} \cdot \hat{\theta} / \kappa$, derived from the assumption of a market price of risk dependent on the spot rate process with $\lambda(t, r(t)) = \lambda_0 \cdot r(t)$ (see Brigo and Mercurio, 2007).

11 Cox, Ingersoll, and Ross (1985) set $c = 2 \cdot \kappa \Big/ \left(\sigma^2 \cdot \left(1 - e^{-\kappa \cdot (t-s)}\right)\right)$, $u = c \cdot r(s) \cdot e^{-\kappa \cdot (t-s)}$, $v = c \cdot r(t)$ and $q = 2 \cdot \kappa \cdot \theta / \sigma^2 - 1$.

$$A(t,h)=\frac{2\cdot\kappa\cdot\theta}{\sigma_r^2}\cdot ln\left(\frac{2\cdot a\cdot e^{(\kappa+a)\cdot\frac{h-t}{2}}}{(\kappa+a)\cdot\left(e^{a\cdot(h-t)}-1\right)+2\cdot a}\right) \text{ and } B(t,h)=\frac{2\cdot\left(e^{a\cdot(h-t)}-1\right)}{(\kappa+a)\cdot\left(e^{a\cdot(h-t)}-1\right)+2\cdot a}$$

with $a=\sqrt{\kappa^2+2\cdot\sigma_r^2}$ (see Björk, 2009).

In the case of the Cox, Ingersoll, and Ross (1985) model, the market price of risk $\lambda(t,r(t))$ is derived from $\lambda(t,r(t))=\lambda_0\cdot\sqrt{r(t)}$, where the market price of risk is proportional to the short rate, rather constant (see Brigo and Mercurio, 2007). Hence, the risk-neutral short rate process is transformed to the real probability world (under $\mathbb{P}$) by setting

$$\begin{aligned} dr(t) &= \left(\kappa\cdot\theta-(\kappa-\lambda_0\cdot\sigma)\cdot r(t)\right)dt+\sigma\cdot\sqrt{r(t)}dW_r^{\mathbb{P}}(t) \\ &= \hat{\kappa}\cdot\left(\hat{\theta}-r(t)\right)dt+\sigma\cdot\sqrt{r(t)}dW_r^{\mathbb{P}}(t). \end{aligned}$$

Hence, the equations $\kappa=\hat{\kappa}-\lambda_0\cdot\sigma$ and $\theta=\hat{\kappa}\cdot\hat{\theta}/\kappa$ demonstrate the relationship between the parameters in an arbitrage free world and the real world probability.

2.4 Maximum likelihood estimation for processes with the property of mean reversion

The calibration of the underlying interest rate model to market data is an important part in the context of quantifying interest rate risk. In the literature exits various methods to estimate the models' input parameters. In the following, the commonly used maximum likelihood estimation approach, a parametric estimation method, is introduced. The basic principle of the (exact) maximum likelihood method is to find the unknown values of the likelihood function that induce the greatest probability, i.e. the values that maximize the likelihood. In the following, we focus on the Vasicek (1977) and Cox, Ingersoll, and Ross (1985) models, both featuring the property of mean reversion in the drift function of the process. First of all, the processes have to be rewritten by[12]

$$dr(t)=\hat{\kappa}\cdot\left(\hat{\theta}-r(t)\right)dt+\sigma dW_r^{\mathbb{P}}(t)=\left(b-a\cdot r(t)\right)dt+\sigma dW_r^{\mathbb{P}}(t) \quad (12)$$

for Vasicek (1977) and

$$dr(t)=\hat{\kappa}\cdot\left(\hat{\theta}-r(t)\right)dt+\sigma\cdot\sqrt{r(t)}dW_r^{\mathbb{P}}(t)=\left(b-a\cdot r(t)\right)dt+\sigma\cdot\sqrt{r(t)}dW_r^{\mathbb{P}}(t) \quad (13)$$

[12] We estimate the dynamic under the real probability measure $\mathbb{P}$.

for Cox, Ingersoll, and Ross (1985) with $a = \hat{\kappa}$ and $b = \hat{\kappa} \cdot \hat{\theta}$.

Vasicek (1977)

To calibrate the Vasicek (1997) model with maximum likelihood estimation, the process in Equation (12) has to be considered in a discrete representation at time t_i, such that[13]

$$r(t_i) = \frac{b}{a} \cdot \left(1 - e^{-a \cdot \Delta t}\right) + e^{-a \cdot \Delta t} \cdot r(t_{i-1}) + \sigma \sqrt{\left(1 - e^{-2 \cdot a \cdot \Delta t}\right) / 2 \cdot a} \cdot Z(t_i),$$

where $Z(t_i)$ denotes a standard normal random number (white noise) (see Brigo and Mercurio, 2007). The time step is given by $\Delta t = t_i - t_{i-1}$ with a total of n observed data points. Now, setting the coefficient to[14]

$$c = e^{-a \cdot \Delta t}, \ d = \frac{b}{a} \cdot \left(1 - e^{-a \cdot \Delta t}\right) \text{ and } V = \sigma \sqrt{\left(1 - e^{-2 \cdot a \cdot \Delta t}\right) / 2 \cdot a},$$

the unknown parameters can be obtained by an OLS regression.[15] Under the assumption of $g = b/a$, the estimates are defined as (see Brigo et al., 2009)

$$\hat{c} = \frac{n \cdot \sum_{i}^{n} r(t_i) \cdot r(t_{i-1}) - \sum_{i}^{n} r(t_i) \cdot \sum_{i}^{n} r(t_{i-1})}{n \cdot \sum_{i=1}^{n} r^2(t_{i-1}) - \left(\sum_{i}^{n} r(t_{i-1})\right)^2}, \ \hat{g} = \frac{\sum_{i=1}^{n} \left(r(t_i) - \hat{c} \cdot r(t_{i-1})\right)}{n \cdot (1 - \hat{c})} \text{ and}$$

$$\hat{V}^2 = n^{-1} \cdot \sum_{i=1}^{n} \left(r(t_i) - \hat{c} \cdot r(t_{i-1}) - \hat{g} \cdot (1 - \hat{c})\right)^2.$$

Cox, Ingersoll, and Ross (1985)

The conditional density $p\left(r(t_i) \middle| r(t_{i-1})\right)$ of the Cox, Ingersoll, and Ross (1985) model (see Equation (13)) is defined by a non-central chi-squared distribution determined by

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13 The discretization is based on the solution of the stochastic differential equation from Vasicek (1977) with $r(t_i) = b/a \cdot \left(1 - e^{-a \cdot (t_i - t_{i-1})}\right) + e^{-a \cdot (t_i - t_{i-1})} \cdot r(t_{i-1}) + \sigma \cdot e^{-a \cdot t_i} \int_{t_{i-1}}^{t_i} e^{a \cdot s} dW(s)$ (see Glasserman, 2004).

14 Applying the Euler discretization scheme, the parameter V simplifies to $V = \sigma \sqrt{\Delta t}$ (see Glasserman, 2004).

15 The OLS (ordinary least squares) method is an approach to calibrate unknown parameters by a linear regression. When the distribution of the underlying process is Gaussian, this method is in accordance with the maximum likelihood estimates (see Brigo et al., 2009). This is the fact in model from Vasicek (1977), where $r(t)$ is (conditional) normal distributed: $r(t) \middle| \mathcal{F}_s \sim \mathcal{N}\left(r(s) \cdot e^{-\kappa \cdot (t-s)} + \theta \cdot \left(1 - e^{-\kappa \cdot (t-s)}\right), \sigma^2 / (2 \cdot \kappa) \cdot \left(1 - e^{-2 \cdot \kappa \cdot (t-s)}\right)\right)$.

$$p\left(r\left(t_i\right)\middle|r\left(t_{i-1}\right)\right)=k\cdot e^{-u-v}\cdot\left(\frac{v}{u}\right)^{q/2}\cdot I_q\left(2\cdot\sqrt{u\cdot v}\right)$$

with $k=2\cdot a\big/\left(\sigma^2\cdot\left(1-e^{-a\Delta t}\right)\right)$, $u=k\cdot r\left(t_i\right)\cdot e^{-a\Delta t}$, $v=k\cdot r\left(t_{i-1}\right)$, $q=2\cdot b\big/\sigma^2-1$ and modified Bessel function I_q.[16] Since the model by Cox, Ingersoll, and Ross (1985) is a Markov process, i.e. it only depends on the previous value, the likelihood function L for calibrating the unknown parameters $\xi=\left(a,b,\sigma\right)$ can be exhibited by

$$L\left(\xi\right)=\prod_{i=1}^{n}p\left(r\left(t_i\right)\middle|r\left(t_{i-1}\right);\xi\right)$$

such that the maximum likelihood estimator,

$$\underset{\xi>0}{\arg\max}\left(\log\left(L\left(\xi\right)\right)\right),$$

is defined (see Brigo et al., 2009).[17]

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16 The modified Bessel function for $z\in\mathbb{R}$ is defines as $I_q\left(z\right)=\sum_{k=0}^{\infty}\left(z/2\right)^{2\cdot k+q}\cdot\left(k!\Gamma\left(k+q+1\right)\right)^{-1}$ (see Iacus, 2010).

17 When the distribution of the underlying process is unknown or analytically difficult to implement, the parameters can i.e. be estimated by simple (explicit) estimation functions or martingale estimating functions. In both approaches, the density of the likelihood function $p\left(r\left(t_i\right)\middle|r\left(t_{i-1}\right);\xi\right)$ has to be approximated. Applying the simple estimation function, the density is substituted for the density of the normal distribution, where mean and variance is defined by the conditional moments of the underlying process. As a consequence of approximating the first two moments, the estimator is biased. However, for sufficient small Δt, the estimator presents reasonable calibration results (see Sørensen, 1997). Unbiased results can be achieved by martingale estimating methods. Here, the estimation function conforms a martingale (and unbiased function) when the unknown parameters $\xi>0$ are the true values (see Iacus, 2010). The estimation function is defined by function $g\left(\Delta t,r\left(t_i\right),r\left(t_{i-1}\right);\xi\right)$ such that $g\left(\Delta t,r\left(t_i\right),r\left(t_{i-1}\right);\xi\right)=\sum_{j=1}^{M}\alpha_j\left(\Delta t,r\left(t_i\right);\xi\right)\cdot h_j\left(\Delta t,r\left(t_i\right),r\left(t_{i-1}\right);\xi\right)$, where $h_j\left(\Delta t,r\left(t_i\right),r\left(t_{i-1}\right);\xi\right)$, $j=1,\ldots,M$, are functions that satisfy the condition $\int g\left(r\left(t_i\right)\middle|r\left(t_{i-1}\right);\xi\right)p\left(r\left(t_i\right)\middle|r\left(t_{i-1}\right);\xi\right)dr\left(t_{i-1}\right)=0$ to determine the martingale estimation function (see Sørensen, 1997). Since the distribution function is usually know known, it is appropriate to use polynomial martingale estimating functions base on the knowledge of the first two conditional moments instead of the underlying density function (see Iacus, 2010)

3. Model Framework

3.1 Partial internal approach

To quantify the market risk of fixed income corporate and government bonds, we apply the approach of Gatzert and Martin (2012) and concentrate on the two main risk drivers: the interest rate risk and the credit risk. The interest rate risk contains the risks arising from the stochasticity of the interest term structure, while credit risk includes the risk of default and changes of the credit spread (over the risk-free interest rate term structure) for bond investments, caused by rating transitions, the variability of the spread and default.

Regarding internal models in supervisory system Solvency II, the SCR for market risk of (corporate and government) bonds assets, $SCR_{mkt,B}^{IM}$, defines the required risk capital needed to cover potential losses corresponding to the Value at Risk of the basic own funds. The time horizon is accounted for one year and the confidence level is set to 99.5% (α = 0.5%). With $MV(t)$ denoting the market value of bond exposures at time t, the SCR for the internal model is defined by[18]

$$SCR_{mkt,B}^{IM} = VaR_{0.005}\left(MV(0) - e^{-\int_0^1 r(t)dt} \cdot MV(1)\right) = MV(0) - VaR_{0.005}\left(e^{-\int_0^1 r(t)dt} \cdot MV(1)\right) \tag{14}$$

with risk-free interest rate $r(t)$, given by the corresponding short rate model.[19]

Regarding credit risk modeling, Jarrow, Lando, and Turnbull (1997) introduce a reduced-form credit risk model, based on credit transition, that includes credit risk (and spread risk) for bonds.[20] First, the credit transition of bond exposures are assumed to follow a Markov chain X in discrete time,

$$X = \left(x(t), t \in \mathbb{N}_0\right), \tag{15}$$

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[18] The integral $\int_0^1 r(t)dt$ is approximated by numerical integration methods (composite trapezoidal rule of the Newton-Cotes formulas, see Press et al., 2007) based on the underlying short rate process $r(t)$.

[19] Christiansen and Niemeyer (2012) discuss alternative definitions of the SCR and the discounting factor in Solvency II.

[20] While Jarrow and Turnbull (2000) emphasize the intrinsic relation between market and credit risk attested by the economic theory and empirical evidence, Brigo and Pallavicini (2007) discuss an alternative credit risk model that takes dependencies between credit and interest rate risk into account. However, we follow regulators and practitioners with the assumption of independency (according to Gatzert and Martin, 2012) since the difficult parameter and correlation calibration is prone to misestimation.

with discrete state space $E = \{1,\ldots k\}$ on the probability space $(\Omega_x, \mathcal{F}_x, \mathbb{Q})$ under the assumption of a complete and arbitrage-free market. The distribution function $(\Psi_x)_{x\in E}$ with

$$\Psi(t,h) = \begin{pmatrix} \psi_{1,1}(t,h) & \cdots & \cdots & \psi_{1,k}(t,h) \\ \vdots & & & \vdots \\ \psi_{k-1,1}(t,h) & \cdots & \cdots & \psi_{k-1,k}(t,h) \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{16}$$

represents the transition probabilities $(\psi_{i,j}(t,h))_{i,j\in E}$, satisfying the conditions $\psi_{i,j}(t,h) \geq 0$, $i \neq j$, and $\psi_{i,i}(t,h) = 1 - \sum_{\substack{j=1 \\ i\neq j}}^{k} \psi_{i,j}(t,h)$. The transition probability corresponds to attaining state *j* at time *h* when starting at state *i* at time *t*. Moreover, the state *k* of the transition matrix in Equation (16) is absorbent and describes the case of default defined by stopping time τ:[21]

$$\tau = \inf\{t \in \mathbb{N} : x(t) = k\}.$$

To obtain risk-neutral transition probabilities $(\psi_{i,j}(t,h))_{i,j\in E}$ that ensure an arbitrage-free market, Jarrow, Lando, and Turnbull (1997) adjust the real world transitions probabilities $(\hat{\psi}_{i,j}(t,h))_{i,j\in E}$ by a time and rating dependent risk premium $\pi_{x(t)=i}(t)$ by[22]

$$\Psi(t,h) = \Pi(t) \cdot \left(\tilde{\Psi}(t,h) - \mathrm{I}\right) + \mathrm{I}$$

with a $k \times k$ matrix $\Pi(t) = \operatorname{diag}\left(\pi_{x(t)=1}(t), \ldots, \pi_{x(t)=k-1}(t), 1\right)$ and a $k \times k$ identity matrix I.[23] Furthermore, the real world transition matrix is assumed to be a time-homogenous Markov chain, exhibited by (see McNeil, Frey, and Embrechts, 2005)

$$\tilde{\Psi}(t,t+1) = \left(\tilde{\psi}_{i,j}(t,t+1)\right)_{i,j\in E} = \left(\tilde{\psi}_{i,j}\right)_{i,j\in E} = \tilde{\Psi}.$$

Jarrow, Lando, and Turnbull (1997) provide the existence of defaultable and non-defaultable zero coupon bonds for all maturities and introduce a deterministic and exogenous given fraction of a non-defaultable bond paid out at maturity *h*, the recovery rate (of treasury) δ, in their

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[21] The process *X* (see Equation (13)) is adapted by filtration $\mathcal{F}_\tau = (\mathcal{F}_{\tau,t})_{t\in\mathbb{N}_0}$.

[22] The risk premium $\pi_{x(t)=i}(t)$ is assumed to be a deterministic function independent of state *j*.

[23] The single entries of matrix $\Psi(t,h)$ can be written as $\psi_{i,j}(t,t+1) = \pi_{x(t)=i}(t) \cdot \tilde{\psi}_{i,j}$, $i \neq j$, and $\psi_{i,j}(t,t+1) = 1 - \pi_{x(t)=i}(t) \cdot (1 - \tilde{\psi}_{i,j})$, $i = j$, where the second constraint ($i = j$) ensures a row sum of one in the risk-neutral distribution $\Psi(t,h)$.

credit risk model.[24] Based on the face value, the recovery rate specifies the amount of redemption to the investors in the case of default.

To take the risk resulting from the uncertainty of the interest term structure into account, we revert to the price process of a non-defaultable zero coupon bond $p(t,h)$ at time t and maturity $h-t$ with $r(t)$ given by an affine term structure model in an arbitrage-free world as in Equation (6) with

$$p(t,h) = E_t^{\mathbb{Q}}\left(e^{-\int_t^h r(s)ds}\right) = e^{A(t,h)-B(t,h)\cdot r(t)}.$$

To extend the price process for a non-default zero coupon bond to the defaultable case, the default and transition process, defined by the Markov chain in Equation (15), has to be integrated in the valuation approach. Following Jarrow, Lando, and Turnbull (1997) and the assumption of independence between the short rate and the credit transition process, accordingly, the price of a defaultable zero coupon bond $\hat{p}_{x(t)=i}(t,h)$ with rating $x(t) = i$ is given by

$$\begin{aligned}\hat{p}(t,h)_{x(t)=i} &= E_t^{\mathbb{Q}}\left(\mathbb{I}_{\{\tau>h\}}\cdot e^{-\int_t^h r(s)ds} + \mathbb{I}_{\{\tau\leq h\}}\cdot\delta\cdot e^{-\int_t^h r(s)ds}\right) = E_t^{\mathbb{Q}}\left(e^{-\int_t^h r(s)ds}\cdot\left(\mathbb{I}_{\{\tau>h\}} + \mathbb{I}_{\{\tau\leq h\}}\cdot\delta\right)\right) \\ &= p(t,h)\cdot\left(\delta + (1-\delta)\cdot\left(1-\psi_{i,k}(t,h)\right)\right),\end{aligned} \tag{17}$$

where $\mathbb{I}_{\{\tau\leq h\}}$ represents the indicator function, which is equal to one if a default occurs until time h and zero otherwise. The expression $1-\psi_{i,k}(t,h)$ denotes the probability for non-defaulting from time t to h.[25] Due to the valuation of defaultable zero coupon bonds with the credit risk model of Jarrow, Lando, and Turnbull (1997) in Equation (17), the price of a defaultable fixed income bond exposure j at time t, $B_j(t)$, can be calculated as the sum of the cash flows $CF_j(h)$, multiplied with the defaultable zero coupon bond prices (see Björk, 2009):

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[24] The model from Jarrow, Lando and Turnbull (1997) implies a recovery of treasury value depending on the seniority. In our model, we assume a recovery rate δ, that is constant for all bond investments.

[25] To avoid the reverting to transition rates (e.g. published by rating agencies) for pricing defaultable zero coupon bonds as in the rating-based Jarrow, Lando, and Turnbull (1997) model, the credit risk model from Lando (1998) or Duffie and Singleton (1999) can be applied, alternatively. Here, the default event is modeled by a Poisson process with an intensity that can be interpreted as a credit spread (see Brigo and Mercurio, 2007). The intensity (or hazard rate) of the issuer can be directly estimated by credit default swaps (CDS) rates (in a time-homogenous or time-inhomogeneous framework) that defines the exposures' credit spread (instead of transition rates).

$$B_j(t) = \sum_{h=t}^{T_j} CF_j(h) \cdot \hat{p}_{x(t)=i}(t,h). \tag{18}$$

The cash flows $CF_j(t)$ of a bond exposure j with maturity $T_j = \max\{t \mid CF_j(t) \neq 0\}$ are determined by the annual coupon payments $C_j(t)$, the face value FV_j, and the number of bonds n_j by

$$CF_j(t) = \begin{cases} C_j(t) \cdot FV_j \cdot n_j & , t < T_j \\ (1 + C_j(t)) \cdot FV_j \cdot n_j & , t = T_j. \end{cases} \tag{19}$$

The number of bond j depends on the invested capital $A_{B,j}(0)$ in bond exposure j at time t = 0. Thus, we set the face value to one, FV_j = 1, and determine the number of bonds by

$$n_j = \frac{A_{B,j}(0)}{B_j^{FV=1}(0)},$$

where $B_j^{FV=1}(0)$ denotes the price of the bond with FV_j = 1 at time t = 0.

For a portfolio consisting of N_B bonds, the stochastic market value $MV_B(1)$ at time t = 1 (without reinvestment) is exhibited by

$$MV_B(1) = \sum_{j=1}^{N_B} \left(\mathbb{I}_{\{\tau>1\}} \cdot \left(B_j(1) + CF_j(1) \right) + \mathbb{I}_{\{\tau\leq 1\}} \cdot \delta \cdot FV_j \cdot n_j \right).$$

3.2 Solvency II standard approach

The SCR calculation for market risk in the standard approach of Solvency II is based on predefined shock scenarios, calibrated on the Value at Risk at a 99.5% confidence level, to cover the variation of the market value of financial instruments (mark to market approach).[26] The time horizon is set to one year (see EIOPA, 2010a, p. 92; European Parliament and the Council, 2009, Article 101, No. 3). In Solvency II, the market risk module determines the SCR arising from capital investments.[27] With respect to the market risk of corporate and govern-

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26 The basis to calculate the SCR under the Solvency II standard model is the change of the net asset value (ΔNAV). When focusing on the asset side (A) only, we keep the liabilities (L) constant: $\Delta NAV = \max\left(A - (A|shock), 0\right)$, i.e. the difference between assets A and liabilities L.

27 Besides the market risk module there exist five further modules in the standard model of Solvency II to determe the basic SCR (BSCR): life, non-life, health, default and the intangibles module. In addition, each

ment bond exposures and the involved interest rate and credit risk, Solvency II provides two sub-modules of the market risk module: the interest rate sub-module and the spread risk sub-module, comprising credit risk (including spread risk).

SCR in the market risk module and diversification benefits

The standard model is designed as a bottom-up approach. Hence, the SCR in the market risk module SCR_{mkt}^{SII} is given by the isolated SCR for the interest rate risk (*int*) sub-module SCR_{int}^{SII} and the spread risk (*sp*) sub-module SCR_{sp}^{SII}, also taking diversification benefits between the risk modules into account. Dependencies between the single sub-modules are included, using the square-root formula, so that the aggregated SCR in the market risk module is given by

$$SCR_{mkt}^{SII} = \sqrt{\sum_{r,c} CorrSCR_{r,c}^{SII} \cdot SCR_{r}^{SII} \cdot SCR_{c}^{SII}},\ where\ r,c \in \{int, sp\}. \tag{18}$$

The predefined correlation parameters $CorrSCR_{r,c}^{SII}$ for the sub-modules, defined as the pair-wise correlation coefficients, are given in Table 2.

Table 2: Correlations in the market risk module in the Solvency II standard model (see EIOPA, 2010a, pp. 108-109)[28]

	Interest	Spread
Interest	1.00	0
Spread	0	1.00

SCR in the interest rate risk sub-module

The interest rate risk sub-module in the Solvency II framework takes the risk from changes in the term structure of the interest rate into account. To quantify the SCR arising from the interest rate risk of bond investments in this sub-module, two present values have to be calculated. A first present value PV_{int} is given by discounting all future cash flows $CF(t)$, defined in Equation (19), with the risk-free interest rate $r_f(t)$ at time t, published by the European Commission. Following, a second present value PV_{int}^{up} represents a possible upward (*up*) movement of the interest rate term structure, causing losses with respect to the market value of corporate

module is divided into several sub-modules. In the case of the market risk module, there are seven sub-modules to quantify different kind of risks: currency, equity, interest rate, property, spread (including credit risk) and illiquidity premium risk as well as the market risk concentration sub-module.

28 The predefined correlation parameter between the interest rate und spread risk sub-module is linked to the scenario in the interest rate sub-module. In this sub-module, two stress scenarios (upward and downward stress) are distinguished, but when accounting the asset side solely, only the upward scenario is of relevance with $CorrSCR_{int,sp}^{SII} = 0$.

and government bonds.[29] In the case of the stressed present value, the risk-free interest rate is shocked by an upward stress $s^{up}(t)$ depending on time t. Thus, the two present values are given by

$$PV_{int} = \sum_{t=1}^{T} \frac{CF(t)}{\left(1+r_f(t)\right)^t} \text{ and } PV_{int}^{up} = \sum_{t=1}^{T} \frac{CF(t)}{\left(1+r_f(t)\cdot\left(1+s^{up}(t)\right)\right)^t}$$

with $T = \max\left(t \mid CF(t) \neq 0\right)$ and the cash flow $CF(t) = \sum_j CF_j(t)$ as the sum of all singe exposures j at time t. Finally, the SCR in the interest rate risk sub-module SCR_{int}^{SII} is given by the difference in the present value and the stressed present value:

$$SCR_{int}^{SII} = \max\left(PV_{int} - PV_{int}^{up}, 0\right). \tag{21}$$

The stress parameters for the upward movement, predefined in the Solvency II standard model, are exhibited in Table 3 for selected point of times (maturities).[30]

Table 3: Interest rate shock (upward movement) in the Solvency II standard model (see EIOPA, 2010a, p. 111)

Maturity t (years)	Relative change $s^{up}(t)$
1	0.70
2	0.70
3	0.64
…	…
25	0.26
>25	0.25

SCR in the spread risk sub-module

The impact of changes of the credit spread (over the risk-free interest rate term structure) is quantified in the spread risk module in Solvency II, including credit and spread risk. This sub-module is defined as a rating-based approach and distinguishes between three uncorrelated groups of exposures: bonds, structured credit products and credit derivates. In our model, we

[29] In general, the sub-module for interest rate risk in Solvency II distinguishes two stress scenarios. One scenario quantifies an increase (upward movement) and one a decrease (downward movement) of the interest term structure. Finally the scenario that induces the higher SCR has to be considered. But for observing the asset side only, just the upward stress scenario generates a positive SCR, so the downward stress will be ignored. For further details see Gatzert and Martin (2012).

[30] Furthermore, Solvency II defines an upward stress for maturities less than one year also with a relative change of 0.70 and for maturities larger than 25 years with 0.25.

only consider the SCR for the group of bonds $SCR_{sp,bonds}^{SII}$, where the SCR is calculated based upon the market value $MV_{sp,j}(0)$ for bond exposure *j*. In the standard approach, the SCR for credit risk of a bond portfolio is defined by multiplication of the market value with the modified duration, denoted by $duration_j$, and a rating-specific stress parameter $F^{up}(rating_j)$ for each bond *j*:[31]

$$SCR_{sp,bonds}^{SII} = \max\left(\sum_j MV_{sp,j}(0) \cdot duration_j \cdot F^{up}\left(rating_j\right), 0\right). \tag{22}$$

Furthermore, the modified duration is limited by a lower (floor) and an upper limit (cap) as displayed in Table 4. A special treatment to calculate the credit risk under Solvency II is given to investments in government bonds. The standard model intends no capital requirements for credit risk, issued by a member state of the EEA in its domestic currency or a currency of an EEA country. Moreover, the disregard of credit risk also affects borrowings, irrespective of the bonds' currency, issued by multilateral development banks, international organizations or the European Central Bank. Thus, all other bond exposures are part of the investment class of non-EEA governments (see CEIOPS, 2010).[32] The rating-specific stress parameters, predefined in Solvency II and distinguished between corporate and non-EEA governments bonds, depend on the current credit rating of each bond and are exhibited in Table 4.[33]

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[31] For the modified duration, not exactly defined in the technical specifications of QIS 5, we use the Macaulay (1938) duration modified discounted by the yield to maturity r_{YtM},

$$duration_j = \frac{\sum_{t=1}^{T_{\max}} t \cdot CF_j(t) \cdot \left(1 + r_f(t)\right)^{-t}}{\sum_{t=1}^{T_{\max}} CF_j(t) \cdot \left(1 + r_f(t)\right)^{-t}} \cdot \frac{1}{1 + r_{YtM}}, \; T_{\max} = \max\left(t \mid CF(t) \neq 0\right),$$

where $r_f(t)$ denotes the risk-free interest term structure given by the European Commission. The yield to maturity r_{YtM} of bond exposure *j* is given by solving the equation $PV_{int,j} = \sum_{t=1}^{T_j} CF_j(t) \cdot \left(1 + r_{YtM}\right)^{-t}$, where $PV_{int,j}$ stands for the present value using the risk-free interest rate $r_f(t)$ at time *t*, given by the European Commission as exhibited in Equation (4).

[32] A further exceptional position in this sub-module is dedicated to AAA-rated covered bonds (mortgage and public sector). The requirements for this treatment of bonds are given in the "undertakings for collective investment in transferable securities" from the European Parliament and the Council (see European Parliament and of the Council, 2005, Article 22, No. 4), and allow to reduce the stress parameter from 0.9% to 0.6% with a duration cap of 53 years (see EIOPA, 2010a, pp. 122-123).

[33] If several different credit ratings exist for one bond exposure, the second best rating has to be applied (see EIOPA, 2010a, p. 121).

Table 4: Spread shock for corporates and non-EEA governments in the Solvency II standard model (see EIOPA, 2010a, pp. 122-123)

	Spread shock corporates			Spread shock non-EEA governments		
Rating	F^{up}	Duration Floor	Duration Cap	F^{up}	Duration Floor	Duration Cap
AAA	0.9%	1	36	-	-	-
AA	1.1%	1	29	-	-	-
A	1.4%	1	23	1.1%	1	29
BBB	2.5%	1	13	1.4%	1	23
BB	4.5%	1	10	2.5%	1	13
B or lower	7.5%	1	8	4.5%	1	10
Unrated	3.0%	1	12	3.0%	1	12

Finally, regarding the SCR calculation in the sub-module for credit risk, the market value $MV_{sp,j}(0)$ of bond exposure j conform with the invested capital $A_{B,j}(0)$ at the starting time t = 0 with

$$MV_{sp,j}(0) = A_{B,j}(0).$$

3.3 Risk types and diversification effects

To isolate the effect of interest rate risk and credit risk of bond investments with respect to the SCR calculation in the context of Solvency II, both risk types have to be regarded separately. First, to quantify the type of interest rate risk solely, the defaultable zero coupon bond price in Equation (17) is equalized with non-default price $p(t,h)$, given by a short term interest rate model (see Equation (6)). By eliminating the credit risk by setting $\psi_{i,k}(t,h) = 0,\ \forall i \in \{AAA, AA, \ldots C\},\ \forall t,h \in \mathbb{N}_0$, the zero coupon bond price for interest rate risk $\hat{p}_{int}(t,h)$ is given by setting

$$\hat{p}_{int}(t,h) = p(t,h). \tag{21}$$

When regarding the isolated SCR for the credit (and spread) risk with the internal model, the defaultable zero coupon bond price with the credit risk model from Jarrow, Lando, and Turnbull (1997) in Equation (17) is set to

$$\hat{p}_{sp,\,x(t)=i}(t,h) = \left(\delta + (1-\delta)\cdot\left(1-\psi_{i,k}(t,h)\right)\right), \tag{22}$$

where $\hat{p}_{sp,\,x(t)=i}\left(t,h\right)$ denotes the defaultable bond price for credit risk only. Here, the interest rate risk is disregarded while assuming the interest rate to be zero, so that $p\left(t,h\right)=1,\ \forall t,h\in\mathbb{N}_0$. Thus, the assumptions in terms of the defaultable zero coupon bond price in Equation (21) and (22) implicate bond prices and SCRs, isolated for one risk type. Regarding the Solvency II standard approach, the isolated SCR for interest rate and credit risk is separately quantified in the interest rate risk and spread risk sub-module in Equation (21) and (22).

Based on the isolated SCR for interest rate risk, we identify diversification effects by integrating both risks in one model. In the standard model of Solvency II, diversification benefits between the sub-modules are included in the aggregation process with the square-root formula in Equation (20). In the case of the internal model, both risks are integrated in the pricing process in Equation (17) by taking interest rate and credit risk into consideration. To account for diversification effects, the diversification in both models is defined as

$$div\left(int,sp\right)=\frac{SCR_{int+sp}}{SCR_{int}+SCR_{sp}}-1, \tag{25}$$

where SCR_{int} denotes the SCR arising from interest rate risk, SCR_{sp} the SCR from credit risk and SCR_{int+sp} the SCR from both types of risk.

4. Numerical Results

In the numerical analysis, we present the solvency capital requirements for corporate and government bonds using four different models for interest rate risk also including credit risk (and spread risk): a partial internal approach distinguished between three affine short rate term structure models, Merton (1973), Vasicek (1977) as well as Cox, Ingersoll, and Ross (1985) and the standard model of Solvency II. After presenting the input parameters, the different interest rate term structure and the associated zero coupon bond price, determined by each interest rate process, are shown. In a next step, the impact of the model selection on the SCR is compared for single bonds and portfolio of bonds including model risk with respect to parameter calibration of the short term interest rate models. Finally, we separately analyze the SCR for interest rate risk as a function of coupon payment and maturity as well as credit risk as a function of credit quality and maturity. All numerical results are given in percentage values of the invested capital to the assets. Monte Carlo methods are employed for the SCR calculation in the internal model with 100,000 paths.[34]

4.1 Input parameters

To ensure comparability, annualized parameters for alternative interest rate models are estimated based on a common interest rate time series. The parameters are calibrated on the three month “Euro Interbank Offered Rate” (EURIBOR) regarding monthly data from 01/1999 to 12/2011. Concerning the model of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) with the property of mean reverting, maximum likelihood methods are used (see, e.g. Brigo et al., 2009). The Merton (1973) model is fitted by mean and volatility estimation. The input data for the short rate models are summarized in Table 5 with speed of the mean reversion $\hat{\kappa}$, long-term mean $\hat{\theta}$ and volatility σ.

Table 5: Estimates of models for the short term interest rate (with annualized parameters)

	$\hat{\kappa}$	$\hat{\theta}$	σ
Merton (1973)	-	-0.0013	0.0069
Vasicek (1977)	0.0950	0.0149	0.0069
Cox, Ingersoll, and Ross (1985)	0.1036	0.0161	0.0390

In line with the predefined risk-free interest rate term structure of Solvency II, we further set the initial value for all short term interest rate models to $r(0) = 0.01$. Furthermore, the market price of (interest rate) risk in the short rate modeling process is assumed to be zero ($\lambda_0 = 0$).

Table 6: Bond portfolio

j	Type	Rating	EEA	Maturity (years)	Coupon p.a. (%)
1	Corporate	AA	-	3	1.250
2	Corporate	AA	-	5	1.375
3	Corporate	AA	-	10	2.950
4	Corporate	B	-	5	10.125
5	Government	AAA	No	3	2.250
6	Government	AAA	No	5	2.750
7	Government	AAA	No	11	3.250
8	Government	B	No	5	6.875

Notes: The data are taken from a database of straight fixed-income bonds, issued in the period range 2/2010 to 5/2011, with B_1, B_2, B_3: Colgate-Palmolive Company, B_4: Air Canada, B_5, B_6, B_7: Canada (Government) and B_8: Ukraine (Government).

[34] In order to ensure stability of the numerical results, we conduct robustness tests by using different sets of sample paths for all results.

In the numerical analysis, eight different fixed income corporate and government bonds are considered. Table 6 displays the bonds' characteristics in detail with type (corporate or government), current credit rating, issued by an EEA member or not, maturity (in years) and annual coupon payment (in %).

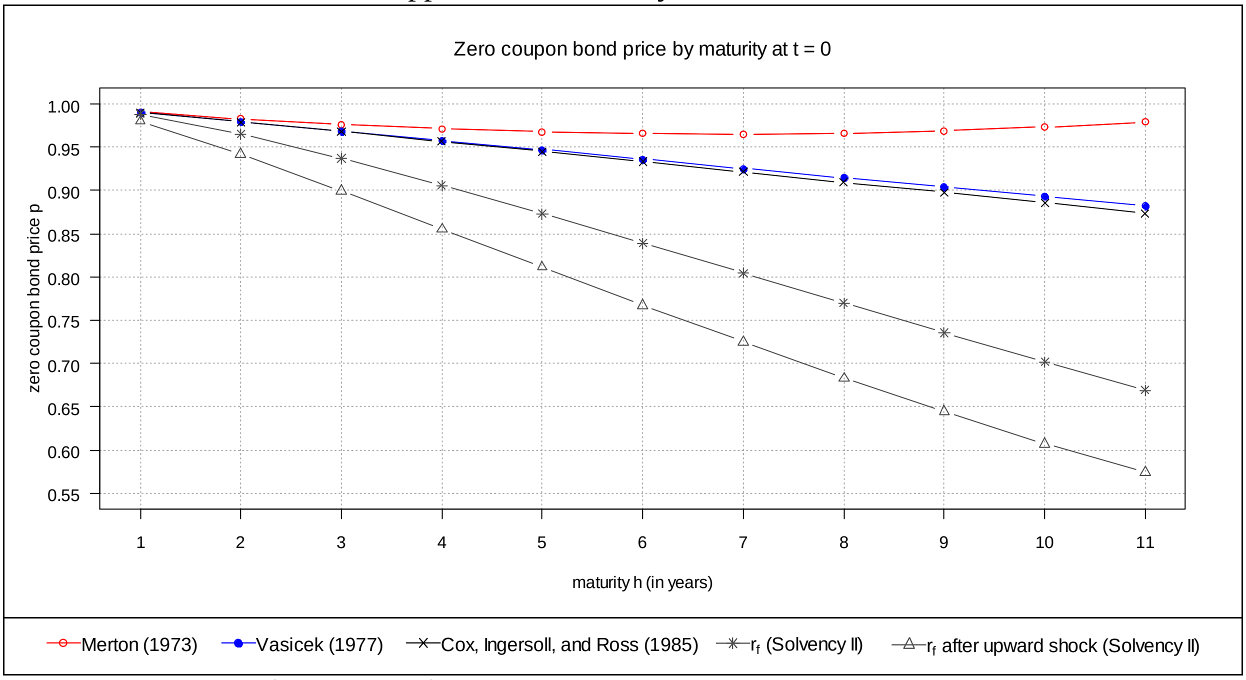
To determine the distribution function of the time-homogenous Markov process in the reduced-form credit risk model from Jarrow, Lando, and Turnbull (1997), we revert to reports published by Standard & Poor's. One report presents the average one-year transition rates for global corporate bonds measured based on bond data from 1981 to 2009 (see Vazza, Aurora, and Kraemer, 2010; Appendix, Table A.1) and another the average one-year transition rates for foreign currency ratings of governments, based on data from 1975 to 2010 (see Chambers, Ontko, and Beers, 2011; Appendix, Table A.2). To deal with corporate and government bonds identified as no longer rated (NR), we follow Bangia et al. (2002). They propose to disregard this information by distributing this rate proportionally to all other rating classes and the default state.[35] In addition, the recovery rate is determined for the case of default in the partial internal approach. We assume a constant recovery rate for all considered corporate and government bonds with $\delta = 0.55$. This assumption is in accordance with the specification of the internal ratings based (IRB) approach to credit risk of Basel II for senior claims on corporate and government bonds not secured by recognized collateral (see BIS, 2006, p. 67).

4.2 Non-defaultable zero coupon prices

First, the risk-free interest rate term structure, given by the four considered models is exhibited in the upper graph in Figure 1. The figure illustrates considerable lower non-defaultable zero coupon prices induced with the model from Vasicek (1977) and Cox, Ingersoll, and Ross (1985) in comparison to Merton (1973). Furthermore, the bond priced induced by the risk-free interest rate term structure of Solvency II and the prices after an upward shock (see Table 3) by time to maturity are exhibited.

[35] Tables A.3 and A.4 in the Appendix represent the transition rates after eliminating the "NR" column with the procedure from Bangia et al. (2002).

Figure 1: Non-defaultable zero coupon bond prices at t = 0 using different affine term structure models and the standard approach of Solvency II



Notes: The input data for the model from Merton (1973), Vasicek (1977) and Cox, Ingersoll, and Ross (1985) are all calibrated on the same data set (see Table 5). Regarding the Solvency II standard model (with and without upward) shock, the zero coupon price (ZCP) is given by $ZCP(t)=\left(1+r_f(t)\right)^{-t}$.

4.3 SCR for single bonds

We next study the SCR for single corporate and government bonds by applying different short term interest rate models for the partial internal model and the Solvency II standard approach. The analyzed bonds are given in Table 6 classified by corporate and governments with varying maturity and credit quality, specified by the credit rating. We quantify the SCR for singles bonds and distinguish different risk sources for bonds: interest rate risk, credit risk and an allowing for of both. Furthermore, diversification benefits are taken into consideration by accounting both risks. In general, the diversification benefit is determined by Equation (25) for both models. In terms of the Solvency II standard model, the diversification arises from the square-root formula while aggregating the market risk module (see Equation (20)).

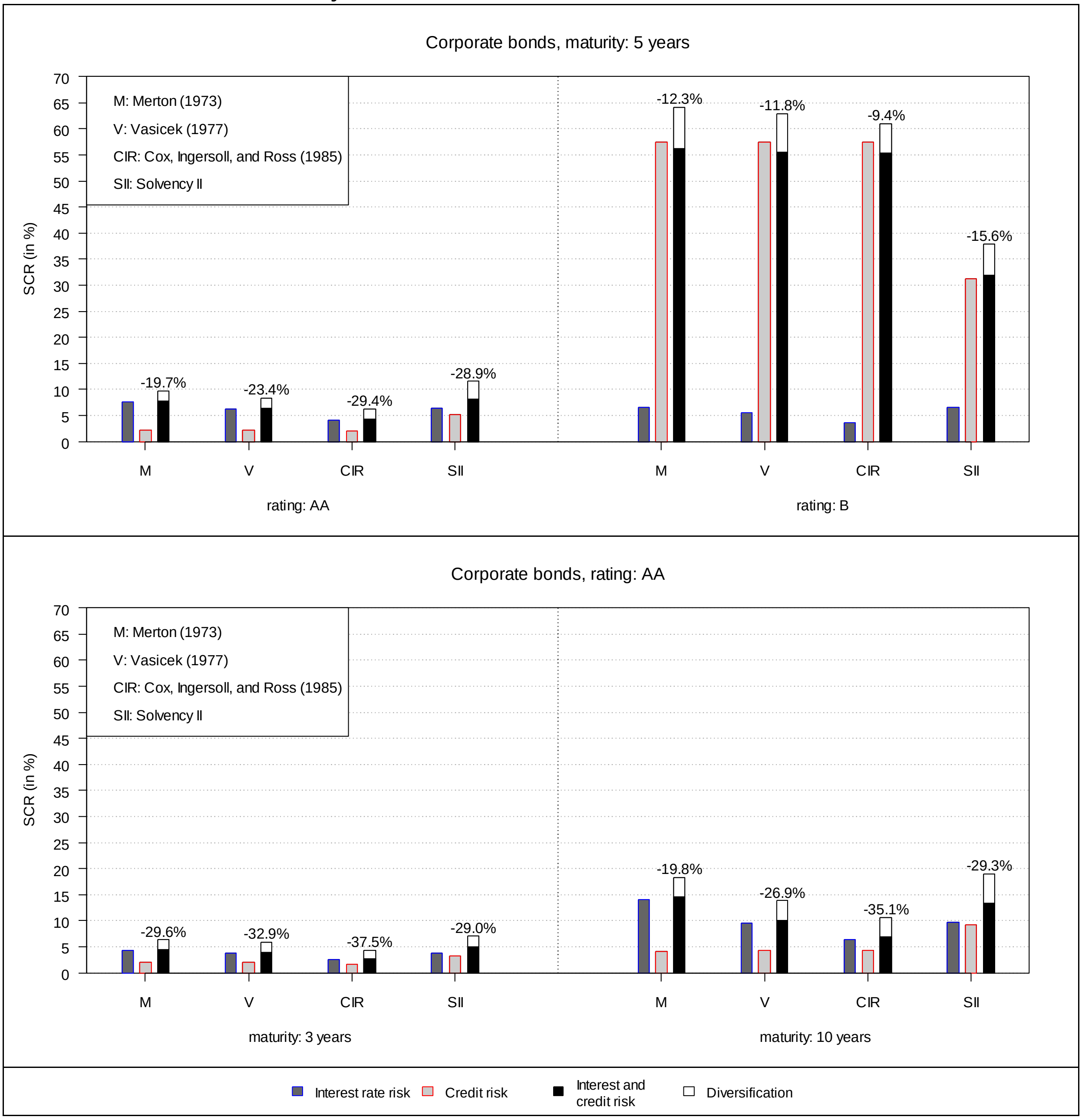
The upper graph in Figure 2 shows the SCR for corporate bonds with a maturity of 5 years and higher (investment grade) as well as lower (speculative grade) credit quality. While the SCR for credit risk is constant for all internal models according to Equation (24), the SCR as a result of the change in the term structure of the interest rate leads to partly considerable different results depending on the short term interest rate model. In the case of the AA-rated bond, the credit risk plays a subordinate role for the SCR calculation for all internal models.

Hence, the interest rate risk and the selection of the short rate model built the essential risk source by calculating the SCR for the higher rated bonds. Regarding the SCR for interest rate risk in the internal model, the model of Merton (1973) leads to the highest SCR, followed by Vasicek (1977) and Cox, Ingersoll, and Ross (1985). The interest rate risk in the standard model is between Vasicek (1977) and Merton (1975) while the credit risk in the standard model implies a higher SCR in comparison to the partial internal approach. For the bond with higher credit quality (AA), the total SCR for both risks is above each internal model. The highest diversification benefit *div*(*int*,*sp*) is achieved by Cox, Ingersoll, and Ross (1985) (-29.4%), however, the absolute diversification (white boxes in Figure 2) in all internal models are approximately equal.

Due to the Value at Risk approach for the SCR in Solvency II (see Equation (14)), the SCR for interest rate risk is particularly affected by the volatility of the underlying short rate process. Hence, the SCR with a partial internal model for interest rate risk increases with increasing volatility of the interest rate process. The approach from Merton (1973) assumes a constant mean and diffusion function, so that the first two moments of the short rate dynamic are boundless, resulting in a high volatility process. Vasicek (1977) also introduces a short rate model with constant diffusion function, but the volatility of the short rate dynamic is restricted by the property of mean reversion in the drift function. A further extension is given in the model by Cox, Ingersoll, and Ross (1985). Besides the mean reversion process as in the case of Vasicek (1977), the diffusion function is extended by the root function of the short rate. Hence, the volatility of the short rate process is even more reduced by Cox, Ingersoll, and Ross (1985) causing the lowest SCR of all considered models (see Figure 2).

Furthermore, the SCR for a corporate bond with lower credit quality and a maturity of 5 years is also given in the upper graph of Figure 2. With respect to the SCR for interest rate risk, the results are quite similar to the higher rated bond. For the investment of the B-rated bonds, the SCR, resulting from changes of the credit spread, builds the key risk driver for the SCR. Compared to the internal model, the Solvency II standard approach strongly underestimates the credit risk and, hence, also the overall SCR. The diversification benefit is higher in the case of the standard approach (-15.6%) and plays an important role to reduce the SCR in all models. In term of the total SCR for both risks, the three internal models approximate each other so that the SCR is nearly independent of the choice of the underlying interest rate process.

Figure 2: Solvency capital requirement for corporate bonds using different affine term structure models and the Solvency II standard model



Notes: Straight fixed-income bonds issued in the period range 2/2010 to 5/2011: Colgate-Palmolive Company (rating (r): AA, maturity (m): 5, coupon (c): 1.375%), Air Canada (r: B, m: 5, c: 10.125%), Colgate-Palmolive Company (r: AA, m: 3, c: 1.25%), Colgate-Palmolive Company (r: AA, m: 10, c: 2.95%).

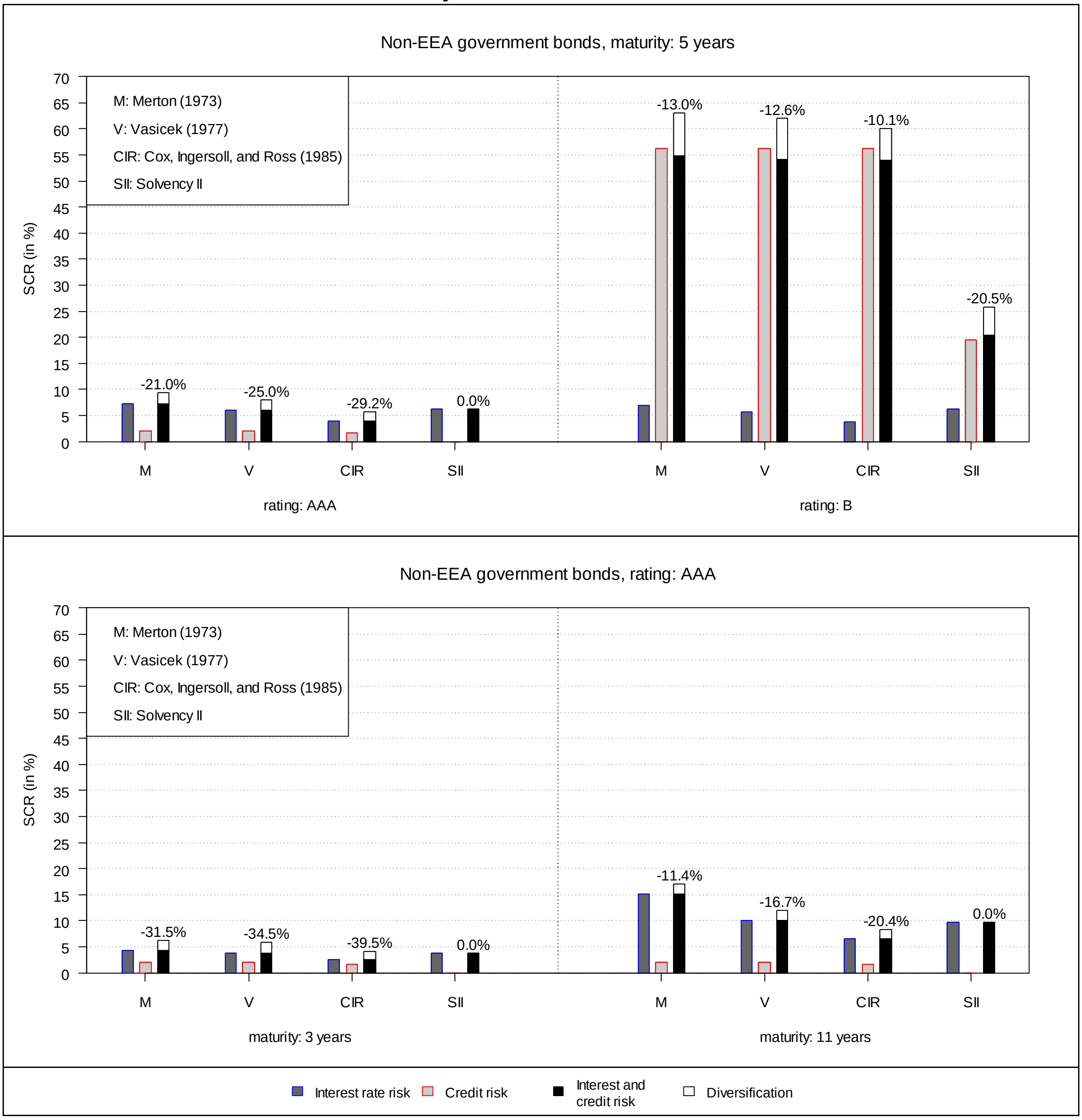
A further analysis, illustrated in the lower graph of Figure 2, reveals the impact of the bonds' maturity by measuring the solvency capital requirement of bond investments. We look at a AA-rated bond with a maturity of three and ten years. As before, the SCR for credit risk is not the main focus for high-rated corporate bonds and cause a higher SCR in the standard approach. However, the interest rate risk increases significantly for bond investments with longer maturity (ten years). Hence, the interest rate risk and accordingly the choice of the short term interest rate model is highly relevant for assets with longer maturities (here ten-years) as a consequence of the increasing interest sensitivity. Here again, the Merton (1973) model provides the highest capital requirements for interest rate and credit risk together followed by the Solvency II standard approach, Vasicek (1977) and Cox, Ingersoll, and Ross (1985). While the Vasicek (1977) and the Solvency II model, in general, tend to similar SCRs for interest rate risk, the highest diversification benefits *div*(*int*,*sp*) provides the application of Cox, Ingersoll, and Ross (1985) with -35.1% and the standard approach (-29.3%).[36]

Next, we focus on the SCR for government bonds issued by countries outside the EEA. Analogously to the corporates in Figure 2, Figure 3 points out the impact of the credit quality on the SCR at constant maturity (lower graph) and the impact of maturity at stable credit rating (upper graph). The results in the upper graph of Figure 3 are similar to the corporate bonds, where the credit risk for bonds with investment grade quality plays a subordinated role. For B-rated (speculative grade) governments, however, the credit risk takes the dominant risk factor for the SCR calculation. Comparable with the corporate bonds, the overall SCR is particularly affected by the interest rate risk and the associated short rate process in the case of high-rated government bonds. With increasing bond maturity, the assets' interest rate sensitivity increases and as a consequence, the differences in the SCR with the internal models diverge even further.

At this point, an important fact are the special regulations for governments, issued by member of the EEA, that do not account for credit risk. The special treatment for the EEA governments implicates a considerable reduction and underestimation of the SCR in the Solvency II standard model for bond with lower credit quality. Hence, the overall SCR consists exclusively of the requirements for interest rate risk.

[36] For the internal models, the diversification given in absolute values approximates each other.

Figure 3: Solvency capital requirement for EEA government bonds using different affine term structure models and the Solvency II standard model

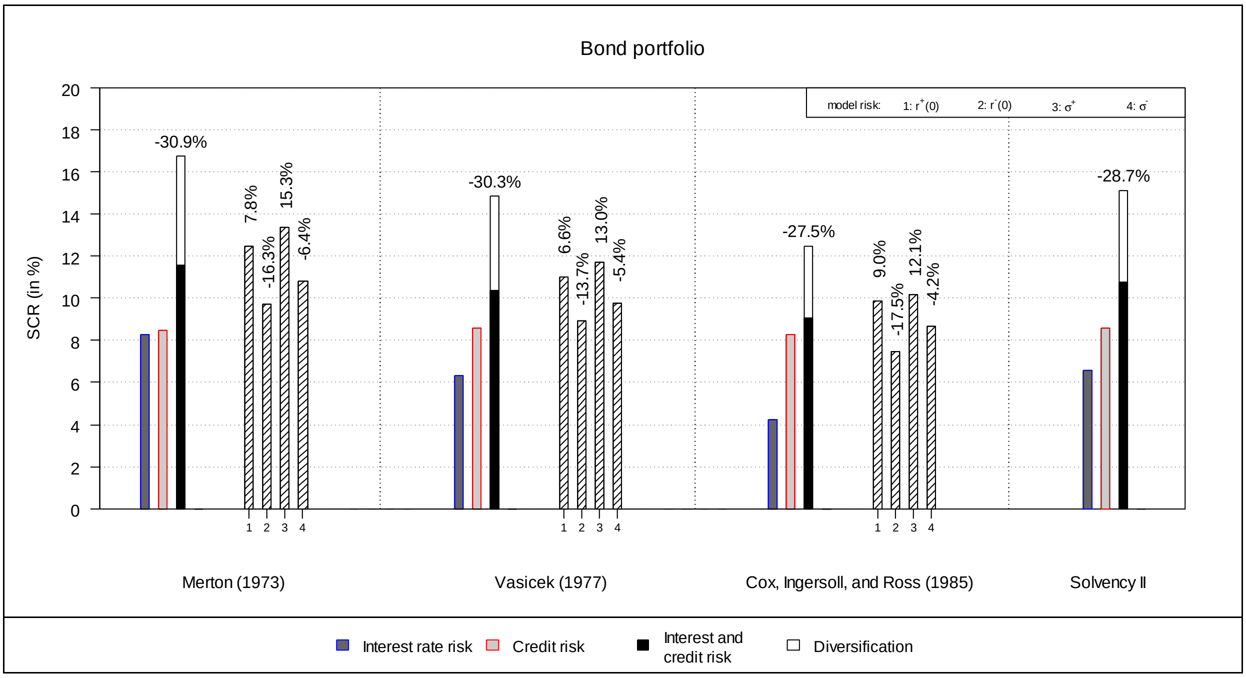


Notes: Straight fixed-income bonds issued in the period range 2/2010 to 5/2011: Canada (Government, rating (r): AAA, maturity (m): 5, coupon (c): 2.75%), Ukraine (Government, r: B, m: 5, c: 6.875%), Canada (Government, r: AAA, m: 3, c: 2.25%), Canada (Government, r: AAA, m: 11, c: 3.25%).

4.4 SCR for bond portfolio and model risk of parameter calibration

After analyzing single bond assets, Figure 6 shows the differences in the SCR calculation for building a portfolio consisting of corporate and governments bonds with equal proportions as given in Table 6. Moreover, the model risk with respect to the input parameter calibration of the short term interest rate models is quantified through shocking the starting value $r(0)$ and the standard deviation σ_r of the interest rate processes by a factor of 20% and -20% separately.

Figure 4: Solvency capital requirement and model risk for a portfolio of bonds (with equal proportions) (see Table 6) using different affine term structure models and the Solvency II standard model



Notes: Straight fixed-income bonds issued in the period range 2/2010 to 5/2011: Colgate-Palmolive Company (rating (r): AA, maturity (m): 3, coupon (c): 1.25%), Colgate-Palmolive Company (r: AA, m: 5, c: 1.375%), Colgate-Palmolive Company (r: AA, m: 10, c: 2.95%), Air Canada (r: B, m: 5, c: 10.125%), Canada (Government, r: AAA, m: 3, c: 2.25%), Canada (Government, r: AAA, m: 5, c: 2.75%), Canada (Government, r: AAA, m: 11, c: 3.25%), Ukraine (Government, r: B, m: 5, c: 6.875%); model risk: +/-20% of original value.

Since the majority of bond investments in the portfolio are based on investment grade credit quality exposures, the credit risk for the portfolio with the internal model falls below the standard model of Solvency II as mentioned while regarding single bond exposures (see Figure 2 and 3). The situation is similar to single bond exposures in the case of the SCR for interest rate risk, where we observe differentiated results depending on the particular interest rate model. The model of Merton (1973) still induce the highest and Cox, Ingersoll, and Ross (1985) the lowest value for the SCR. Regarding the SCR for both risks, credit and interest rate risk, the Solvency II standard approach approximately corresponds to the results of the Vasicek (1977) model. The diversification benefit $div(int,sp)$ in the portfolio case tends to the same level across all four models (with approximately 30%).

With regard to model risk, we assume a separate increasing or decreasing of the initial value $r(0)$ and the short rate process volatility σ_r of 20%. Figure 4 shows that both, the starting value and the volatility, each separately shocked, have strong impact on interest rate risk and are therefore identified as important risk sources, highly prone to errors in the input calibration. In particular, a decrease in the starting value $r(0)$ reduce the SCR up to 17.5% (Cox, Ingersoll, and Ross, 1985) and, moreover, an increase in the standard deviation σ_r of the short rate process raise the required capital up to 15.3% (Merton, 1973).

Table 7: Dimension of over- or underestimation the SCR with the internal model compared to the Solvency II standard model

	Merton (1973)			Vasicek (1977)			Cox, Ingersoll, and Ross (1985)		
	int	*sp*	*int + sp**	*int*	*sp*	*int + sp**	*int*	*sp*	*int + sp**
B_1	19%	-59%	-6%	-3%	-58%	-22%	-35%	-61%	-47%
B_2	0%	84%	76%	-17%	84%	74%	-45%	84%	73%
B_3	13%	-38%	-11%	1%	-36%	-21%	-32%	-47%	-46%
B_4	44%	-55%	8%	-2%	-54%	-25%	-35%	-54%	-49%
B_5	19%	-	19%	-3%	-	-3%	-35%	-	-35%
B_6	10%	189%	168%	-10%	189%	164%	-40%	189%	164%
B_7	13%	-	13%	1%	-	1%	-32%	-	-32%
B_8	56%	-	56%	3%	-	3%	-32%	-	-32%
B_{pf}	26%	-1%	7%	-4%	0%	-4%	-36%	-3%	-16%

** Including diversification benefits.*
Notes: Straight fixed-income bonds issued in the period range 2/2010 to 5/2011: B_1: Colgate-Palmolive Company (rating (r): AA, maturity (m): 3, coupon (c): 1.25%), B_2: Colgate-Palmolive Company (r: AA, m: 5, c: 1.375%), B_3: Colgate-Palmolive Company (r: AA, m: 10, c: 2.95%), B_4: Air Canada (r: B, m: 5, c: 10.125%), B_5: Canada (Government, r: AAA, m: 3, c: 2.25%), B_6: Canada (Government, r: AAA, m: 5, c: 2.75%), B_7: Canada (Government, r: AAA, m: 11, c: 3.25%), B_8: Ukraine (Government, r: B, m: 5, c: 6.875%); B_{pf}: Portfolio of B_1 to B_8. The dimension of over- or underestimating the SCR, denoted by ΔSCR(x), is calculated by setting $\Delta SCR(x) = \left(SCR_x^{IM} - SCR_x^{SII}\right) / SCR_x^{SII}, x \in \{int, sp, int + sp\}$.

Table 7 provides an overview of the dimension of over- or underestimation the SCR with the internal model in comparison to the Solvency II standard approach. The entries in Table 7 reflect the differences (given in percentages) of each model and each risk based on the SCR with the standard model for all considered bonds and a portfolio with equal proportions (see Table 6), following Figure 2, 3 and 4. As before, the internal model is distinguished between the three affine short rate models from Merton (1973), Vasicek (1977) and Cox, Ingersoll, and Ross (1985). Furthermore, the different models distinguish interest rate risk (*int*) and credit risk (including spread risk) (*sp*) separately (see Equation (23) and (24)) as well as both risks together (*int* + *sp*) as given in Equation (17). Regarding the total SCR for interest rate risk and credit risk together, diversification benefits are included. For interest rate risk only, the SCR of the standard model always tends to the results with the Vasicek (1977) model. The model from Merton (1973) induce partly significant higher SCRs up to 56% (see B_8) while the Cox, Ingersoll, and Ross (1985) always calculates the lowest capital requirements up to -40% (see

B_2) The credit risk is considerable underestimated by 84% and 189% in the standard approach in comparison to the internal model for speculative grade corporate and government bonds regardless of the short rate process selection (see B_2 and B_6). In contrast, the SCR for credit risk of investment grade bonds is overestimated with the Solvency II model (see B_1, B_3 and B_4). For AAA- and AA-rated non-EEA governments, the credit risk is set to zero in the standard approach (see B_5, B_7 and B_8).[37] The underestimation of credit risk for low-rated bonds in Solvency II also causes higher values for the total SCR of both risks, interest rate and credit risk. In the case of high-rated assets where the credit risk does not dominate, the Vasicek (1977) model approximately tends up to 1% to the SCRs with the standard approach (see B_1, B_3, B_4, B_5, B_7 and B_8). The internal approach with Merton (1973) induces in the majority of cases higher SCRs than the Solvency approach. However, regarding Cox, Ingersoll, and Ross (1985), the total SCR always lies significantly below the Solvency II model (with the exception of the low-rated bonds B_2 and B_6). Similar results are achieved with a portfolio of the bond assets with equal proportions. In this case, the Vasicek (1977) model approximates the standard approach of Solvency II up to -4% for the total SCR of interest rate and credt risk (B_{pf}). The asset class of EEA-governments have an exception position when quantifying the SCR. As a consequence of disregarding credit risk in the standard model, the credit risk for these investments is always underestimated and significantly underestimated for low-rated bonds.

4.5 SCR for interest rate and credit risk as a function of bonds characteristics

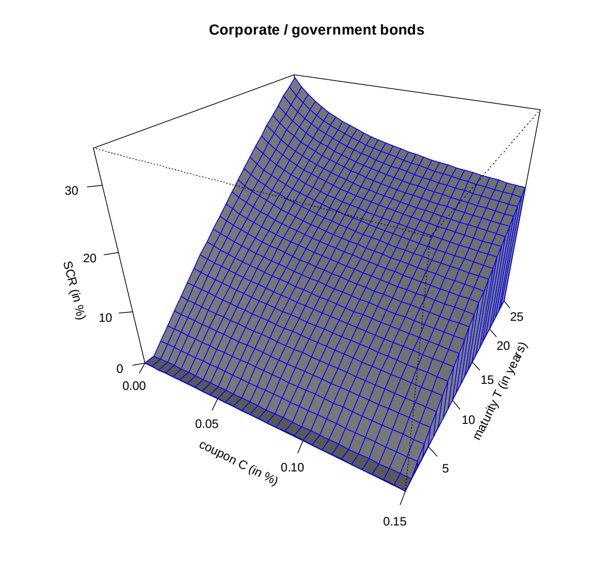
In general, the SCR for interest rate risk of bond investments depends on two asset characteristics, irrespective of the bond type (corporate / government): the (annual) coupon payment and the maturity. For this purpose, we next analyze the result of the solvency capital requirements as a function of coupon and maturity (in years) for all four models.[38] A comparison of the different models, exhibited in Table 5, confirms the previous results for specific single bond exposures (see Figure 2, 3 and 4). Due to the volatility of the underlying short rate process, the model of Merton (1973) in Figure 5 a) generates the highest SCR level, the lowest capital requirements are caused by the Cox, Ingersoll, and Ross (1985) model almost independent of coupon and maturity (see Figure 5 c)). As before, the Solvency II standard model and the approach from Vasicek (1977) approximate each other (see Figure 5 b) and d)).

[37] For B_5, B_7 and B_8, no dimension of over- or underestimation the SCR for credit risk with the internal approach can be measured, since the SCR for credit risk (in the spread risk sub-module) of AAA- and AA-rated non-EEA governments is set to zero in the standard model of Solvency II (see EIOPA, 2010a, p. 123).

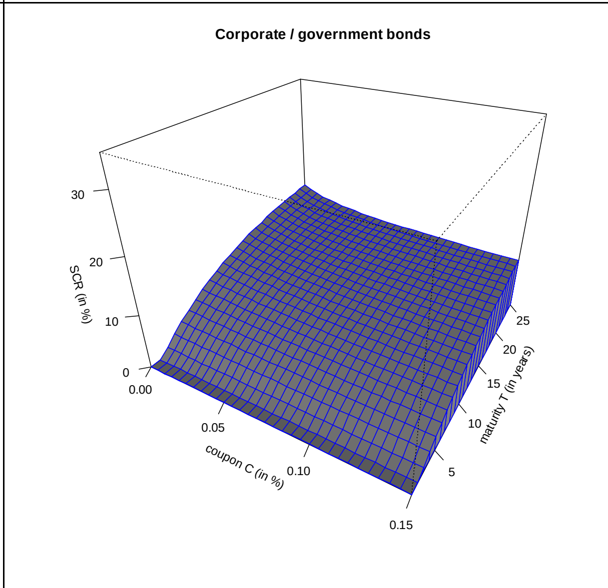
[38] In this section, we conduct a hypothetical analysis to examine the impact of coupon payments, rating and maturity on the SCR.

Figure 5: Solvency capital requirement for interest rate risk as a function of coupon payment and maturity

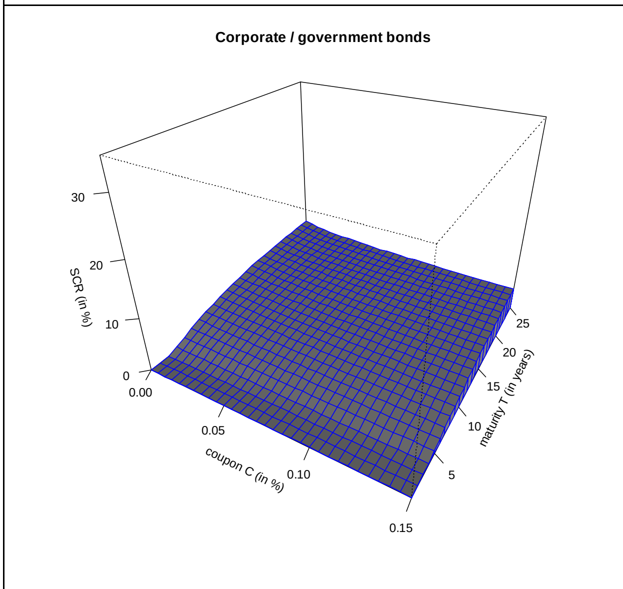
a) Internal model: Merton (1973)



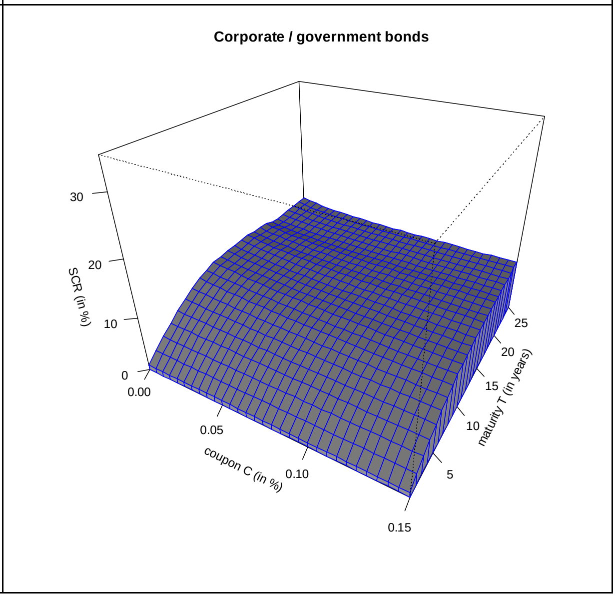
b) Internal model: Vasicek (1977)



c) Internal model: Cox, Ingersoll, and Ross (1985)



d) Solvency II standard model*

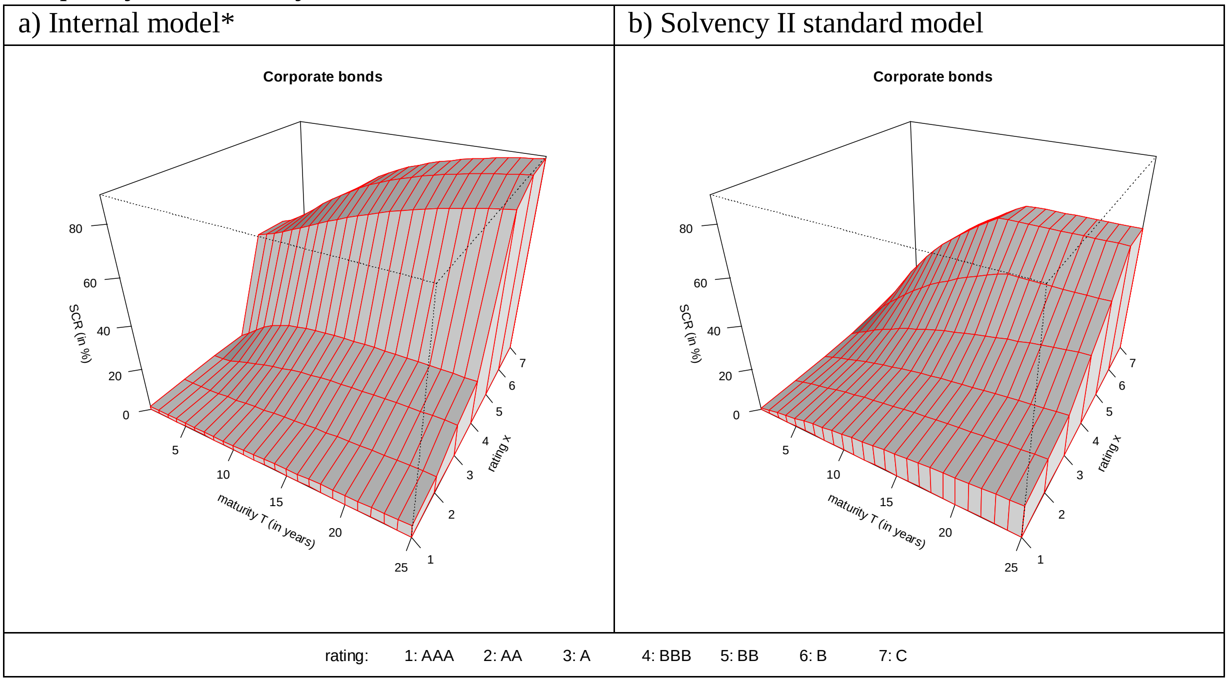


** The face value and, hence, the nominal annual coupon payment in the Solvency II standard model are calibrated with the Cox, Ingersoll, and Ross (1985) model.*

In addition to the risk for changes in the term structure of the interest rate, another important factor for the SCR of bond investments is the credit risk. As shown in Figure 2 und 3, in particular speculative grade bond induce high solvency capital requirements for credit risk in the internal model and, to a more less extent, in the standard model of Solvency II. With respect to bond exposures, the risk drivers for credit risk are identified by the credit quality, the maturity and, to a small extend, the coupon payment. In particular the credit quality and the maturity of the exposure specify the level of risk with respect to SCR. In the quantification of credit risk, the coupon payment plays a subordinated role for the SCR in the internal model (see Equation (18) and (24)) as well as in the standard approach of Solvency II (see Equation (22)).

First, we examine in Figure 6 the SCR for corporate bonds calculated with internal model, based on the rating-based credit risk model from Jarrow, Lando, and Turnbull (1997), and the rating-based Solvency II standard model. Apart from an increasing SCR with an increase in maturity, both models offer differences as a function of the credit rating. Regarding bonds with an investment grade credit quality, the Solvency II standard approach slightly overestimates the SCR in comparison to the internal model. The case is different for speculative grade corporate, where the risk is significantly underestimated in the standard model. Furthermore, with regard to the internal approach, there is a considerable gap in the SCR results by the transition from the investment grade to the speculative grade credit quality (BBB / BB). This is due to the specification of the transition distribution of the Markov process in Equation (16), given in the Appendix, Table A.3, where the downgrade and accordingly the default probability increases significantly for corporate bonds with BB-rating quality or lower.

Figure 6: Solvency capital requirement for credit risk of corporate bonds as a function of credit quality and maturity



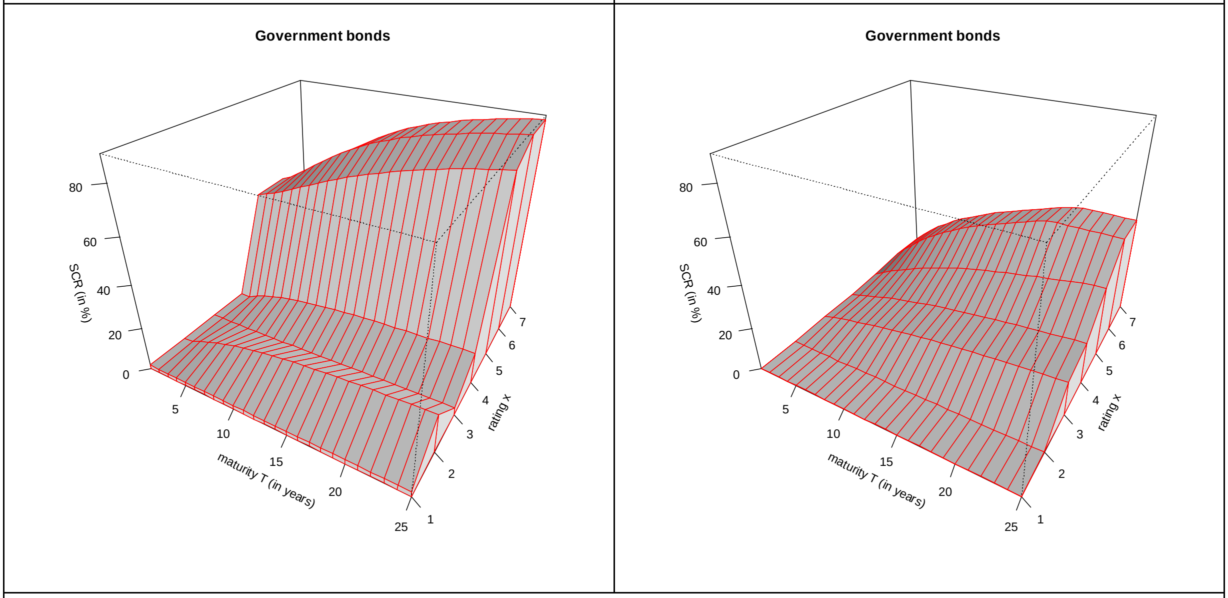
** The discount factor to obtain the SCR (see Equation (14)) is specified by the Cox, Ingersoll, and Ross (1985) model.*

Notes: The coupon payment for corporate bonds cp_c is hypothetical calibrated on rating specific coupons rcp_c and defined as a function of credit rating r and maturity t by $cp_c = rcp_c(r)\cdot\sqrt{t/2}$. The rating specific coupon payments are set to rcp_c(“AAA”) = 0.01, rcp_c(“AA”) = 0.015, rcp_c(“A”) = 0.02, rcp_c(“BBB”) = 0.025, rcp_c(“BB”) = 0.03, rcp_c(“B”) = 0.06 and rcp_c(“C”) = 0.08.

In a second step, we focus on government bonds and analyze the SCR for credit risk with the two approaches, the internal model and the Solvency II standard model. The results in Figure 7 show very similar SCRs as for corporate bonds (see Figure 6). However, in the case of government bonds, the SCR level generally lies below the SCR for corporate. Analog to the corporates, the standard model overestimates the SCR for investment grade bonds and underestimates the credit risk for speculative credit quality governments. The gap in the SCR from BBB to BB-rated bonds in the internal model, i.e. the transition from investment and speculative quality, is explained by the transition rates for government bonds and given in Appendix, Table A.4.

Figure 7: Solvency capital requirement for credit risk of government bonds as a function of credit quality and maturity

a) Internal model*	b) Solvency II standard model



rating: 1: AAA 2: AA 3: A 4: BBB 5: BB 6: B 7: C

** The discount factor to achieve the SCR (see Equation (14)) is specified by the Cox, Ingersoll, and Ross (1985) model.*

Notes: The coupon payment for government bonds cp_g is hypothetical calibrated on rating specific coupons rcp_g and defined as a function of credit rating r and maturity t by $cp_g = rcp_g(r)\cdot\sqrt{t/3}$. The rating specific coupon payments rcp_g are set to rcp_g(“AAA”) = 0.01, rcp_g(“AA”) = 0.015, rcp_g(“A”) = 0.02, rcp_g(“BBB”) = 0.025, rcp_g(“BB”) = 0.03, rcp_g(“B”) = 0.06 and rcp_g(“C”) = 0.08.

5. Conclusions

In this paper, we examine the model risk of measuring the solvency capital requirements for corporate and government bonds with a partial internal model in the context of Solvency II. A special focus in the analysis is laid on interest rate risk, being a significant important position in the solvency capital requirement calculation, in particular for life and health insurers. Three approaches for modeling the interest term structure are considered: the short rate model from

Merton (1973), Vasicek (1977) and Cox, Ingersoll, and Ross (1985), all including the benefit of an affine term structure to valuate interest rate sensitive assets analytically. To obtain comparability, the short rate models for interest are estimated based on a common interest rate time series. In addition to interest rate risk, bond assets are primarily affected by credit risk that is also integrated in the internal approach and modeled by the credit risk approach from Jarrow, Lando, and Turnbull (1997). The credit risk model is based on the assets' credit quality and quantifies the risk by the probability of changes in the credit rating due to default. Hence, this procedure includes credit in the internal approach. Finally, the solvency capital requirements with the internal model and the three interest rate models are compared with the results of the Solvency II standard model for different corporate and government bonds, i.e. different maturity and credit quality. With respect to the standard approach of Solvency II, we analyze the interest rate risk as well as the spread risk sub-module that contains credit risks.

Our results concerning the different short rate models for measuring the SCR for interest rate risk show that the model selection can have a large impact on the SCR for corporate and government bonds. Considerable differences result in particular for high rated bonds and thereby increase with increasing maturity of the exposure. Regarding bond exposures with lower credit quality, the analysis demonstrates a negligibly small impact on the selection of the short rate process and the internal models approximate each other, as a consequence of high requirements for credit risk. For investment grade bonds, however, the short rate model selection plays a significant role for the SCR calculation in the internal model. One main risk driver for interest rate risk is given by the volatility of the underlying short rate process: an increase in the volatility of the interest rate process implicates an increasing SCR for investment grade assets. A comparative analysis of higher rated bond exposures with the internal and the standard model always induces the lowest SCR level using the model from Cox, Ingersoll, and Ross (1985), followed by Vasicek (1977) and Merton (1975). The difference in the SCR can be explained by the varying volatility, even though all models are calibrated on the case data set. In general, the SCR resulting from the standard model of Solvency II is in accordance with the model from Vasicek (1977). Regarding interest rate risk only (without credit risk), the same model order for the SCR applies.

Furthermore, an analysis of single investment grade corporate and government bonds emphasizes the significant model risk for interest rate with respect to short rate models. When comparing the standard model with the internal models for single bond assets, the SCR for interest rate risk vary from a -45% lower SCR (Cox, Ingersoll, and Ross, 1985) up to 56% higher requirements (Merton, 1975). However, for a portfolio of corporate and governments bonds, the fluctuation is reduced to -36% (Cox, Ingersoll, and Ross, 1985) and 26% (Merton, 1975). The model from Vasicek (1977) is approximately in accordance with the requirements given by the Solvency II approach.

Diversification benefits, in general, represent an important aspect for calculating a total SCR for corporate and government bond investments implying interest rate and credit risk (including spread risk). For our examined bonds, the benefit derived by diversification is up to 40% for single bonds and up to 31% for the portfolio of bonds. Thus, diversification is of high importance to the total SCR of bond exposures. For investment grade bonds, the largest diversification benefit is achieved by the Cox, Ingersoll, and Ross (1985) model, however, regarding the diversification in absolute values, the benefit is approximately equal. Beside the model selection, moreover, the input parameters of the short term interest rate models are detected as sensitive to calibration. The starting value and, in particular, the standard deviation are identified as a significant risk driver and are highly prone to errors in the input calibration.

In summary, the measurement of interest rate risk with a partial internal model in the context of Solvency II is affected by model risk in several ways. First and foremost, the selection of the process for modeling the risk-free interest term structure is of relevance. Different short rate models partly deliver diverging results for single bond assets. In our analysis, the model from Cox, Ingersoll, and Ross (1985) always induce the lowest solvency capital requirements compared to the other internal models as well as the Solvency II standard model. Considering a portfolio of bonds the same result remains, while the SCR with the internal models converg for low-rated bonds. Furthermore, to provide an adequate reflection of the insurers' risk situation, the input parameters should be critically analyzed. In addition to interest rate risk, credit risk is of high of importance to low-rated bonds and not sufficiently reflected in the standard model of Solvency II.

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APPENDIX

Table A.1: Global corporates average one-year transition rates (%) (see Vazza, Aurora and Kraemer, 2010, p. 27)

	AAA	AA	A	BBB	BB	B	C	D	NR
AAA	88.21	7.73	0.52	0.06	0.08	0.03	0.06	0.00	3.31
AA	0.56	86.60	8.10	0.55	0.06	0.09	0.02	0.02	4.00
A	0.04	1.95	87.05	5.47	0.40	0.16	0.02	0.08	4.83
BBB	0.01	0.14	3.76	84.16	4.13	0.70	0.16	0.26	6.68
BB	0.02	0.05	0.18	5.17	75.52	7.48	0.79	0.97	9.82
B	0.00	0.04	0.15	0.24	5.43	72.73	4.65	4.93	11.83
C	0.00	0.00	0.21	0.31	0.88	11.28	44.98	27.98	14.36

Table A.2: Governments average one-year transition rates (%) (see Chambers, Ontko and Beers, 2011, p. 41)[39]

	AAA	AA	A	BBB	BB	B	C	D	NR
AAA	97.78	2.22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	3.37	93.64	2.25	0.00	0.37	0.37	0.00	0.00	0.00
A	0.00	3.60	92.80	3.60	0.00	0.00	0.00	0.00	0.00
BBB	0.00	0.00	6.75	89.03	3.38	0.84	0.00	0.00	0.00
BB	0.00	0.00	0.00	6.14	88.06	4.10	1.02	0.68	0.00
B	0.00	0.00	0.00	0.00	7.20	86.36	3.41	1.89	1.14
C	0.00	0.00	0.00	0.00	0.00	31.82	31.82	36.36	0.00

Table A.3: Global corporates average one-year transition rates derived from Table A.1 accounting for non-rated corporates (NR) (%, rounded values)

	AAA	AA	A	BBB	BB	B	C	D
AAA	91.23	7.99	0.54	0.06	0.08	0.03	0.06	0.00
AA	0.58	90.21	8.44	0.57	0.06	0.09	0.02	0.02
A	0.04	2.05	91.47	5.75	0.42	0.17	0.02	0.08
BBB	0.01	0.15	4.03	90.18	4.43	0.75	0.17	0.28
BB	0.02	0.06	0.20	5.73	83.74	8.29	0.88	1.08
B	0.00	0.05	0.17	0.27	6.16	82.49	5.27	5.59
C	0.00	0.00	0.25	0.36	1.03	13.17	52.52	32.67

[39] To obtain row sums equal to one and thus a cumulative distribution function for each row, we adjust individual values in Table A.2.

Table A.4: Governments average one-year transition rates derived from Table A.2 accounting for non-rated governments (NR) (%, rounded values)

	AAA	AA	A	BBB	BB	B	C	D
AAA	97.78	2.22	0.00	0.00	0.00	0.00	0.00	0.00
AA	3.37	93.64	2.25	0.00	0.37	0.37	0.00	0.00
A	0.00	3.60	92.80	3.60	0.00	0.00	0.00	0.00
BBB	0.00	0.00	6.75	89.03	3.38	0.84	0.00	0.00
BB	0.00	0.00	0.00	6.14	88.06	4.10	1.02	0.68
B	0.00	0.00	0.00	0.00	7.28	87.36	3.45	1.91
C	0.00	0.00	0.00	0.00	0.00	31.82	31.82	36.36