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Convergence of Capital and Insurance Markets: Consistent Pricing of Index-Linked Catastrophic Loss Instruments

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CONVERGENCE OF CAPITAL AND INSURANCE MARKETS: CONSISTENT PRICING OF INDEX-LINKED CATASTROPHIC LOSS INSTRUMENTS

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ABSTRACT

Index-linked catastrophic loss instruments have become increasingly attractive for investors and play an important role in risk management. Their payout is tied to the development of an underlying industry loss index (reflecting losses from natural catastrophes) and may additionally depend on the ceding company's loss. Depending on the instrument, pricing is currently not entirely transparent and does not assume a liquid market. We show how arbitrage-free and market-consistent prices for such instruments can be derived by overcoming the crucial point of tradability of the underlying processes. We develop suitable approximation and replication techniques and – based on these – provide explicit pricing formulas using cat bond prices. Finally, we use empirical examples to illustrate the suggested approximations.

Keywords: Alternative risk transfer; cat bonds; industry loss warranties; pricing approaches; risk-neutral valuation.

JEL Classification: G13, G22

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1. INTRODUCTION

Alternative risk transfer (ART) has become increasingly relevant in recent years for insurers and investors,¹ especially due to a considerably growing risk of extreme losses from natural catastrophes caused by value concentration and climate change, as well as the limited (and volatile) capacity of traditional reinsurance markets in the past (Cummins, Doherty and Lo, 2002). In this context, ART intends to provide additional (re)insurance coverage by transferring insurance risks to the capital market. This offers considerably higher capacities and can thus help satisfy the high demand as well as reduce the market power of reinsurance companies (Froot, 2001). Moreover, ART could fix the problem of “nondiversification traps” in the catastrophe insurance markets by offering new risk transfer opportunities besides the pooling of risks in the reinsurance market as described in Ibragimov, Jaffee and Walden (2009). Among the most commonly used ART instruments are index-linked catastrophic loss instruments such as index-based cat bonds² or industry loss warranties (ILWs), for instance, whose defining feature is their dependence on an industry loss index and which may also depend on the company-specific loss resulting from a natural catastrophe.³ However, the current degree of liquidity of the various index-linked instruments considerably differs. While the market for cat bonds is fairly well developed with an increasingly relevant secondary market (Albertini, 2009), for instance, the market for ILWs is less liquid and limited (Elementum Advisors, 2010).

In this paper, we focus on how these products can be priced in a consistent way and discuss under which assumptions (e.g., regarding a liquid underlying market) risk-neutral valuation can be used. This procedure can considerably simplify pricing and enhance transparency, making the market as a whole more efficient. In addition, risk-neutral valuation is of great relevance for the inclusion of such instruments in enterprise risk management strategies as it provides a mark-to-market valuation approach, allowing for (partial) hedging, versus the traditional mark-to-model approaches with the associated model risk (which is very hard to quantify). We develop new pricing approaches by means of approximations and replication techniques and apply them to industry loss warranties (ILWs) as a representative of index-linked catastrophic loss instruments under the assumption of a liquid cat bond and stock market, while carefully addressing the necessary prerequisites and limitations, and we also illustrate the approaches by consistently pricing different cat bonds. We study binary

¹ The volume of outstanding cat bonds, for instance, reached \$17.5bn in 2013 (see AON (2013)). Investors include, e.g., specialized funds, institutional investors, mutual funds, and hedge funds (see AON (2013)).

² There are various versions of cat bonds with different types of triggers, including indemnity-based and non-indemnity based triggers with parametric, modelled loss, and industry loss triggers, for instance (see, e.g., Hagedorn et al. (2009)),

³ See, e.g., Cummins and Weiss (2009) and Barrieu and Albertini (2009) for an overview of the ART market.

contracts in detail, whose payout depends on the industry index only, and discuss indemnity-based contracts, where the payout depends on both the industry index and the individual company losses, thus representing a double-trigger product. The approaches derived in this paper can also be transferred to the consistent pricing of other index-linked catastrophic loss instruments.

In the literature, several papers examine the actuarial and financial pricing of index-linked catastrophic loss instruments such as ILWs, for instance, (e.g., Ishaq (2005), Gatzert and Schmeiser (2012), Braun (2011)) and discuss the underlying assumption briefly (see Braun (2011)). However, the tradability of the underlying processes as well as direct replication and consistent pricing has not been discussed in detail so far in this context. However, several papers have dealt with risk-neutral valuation in the context of cat bonds (see, e.g., Nowak and Romaniuk (2013), Haslip and Kaishev (2010)) and compute explicit pricing formulas, while other authors focused on the consistent pricing of double-trigger contracts (e.g., Lane, 2004) or empirical aspects using econometric pricing approaches (e.g., Jaeger, Müller and Scherling (2010), Galeotti, Guertler and Winkelvos (2012), Braun (2014)).

In general, the main assumption when using risk-neutral valuation is the tradability of an underlying process. Since the underlying process can usually not be traded directly like a stock, one has to assume a liquid market for (certain) derivatives. We derive a general approach for dealing with this issue, describe the underlying assumptions and apply this approach to binary ILWs as well as cat bonds. This is done by means of direct or approximate replication with traded derivatives using available cat bonds, which leads to explicit and consistent prices.⁴ In particular, using ILWs as an example, we assume the existence of a liquid cat bond market to handle the tradability of the industry loss index and to apply arbitrage-free valuation. Since there is a growing secondary market for cat bonds, this assumption appears to be at least appropriate in the foreseeable future (see, e.g., Albertini (2009) for a description of the secondary market). Moreover, we show that liquidity assumptions are not needed to the same extent as in classical option pricing theory because continuous trading is not necessary to replicate ILWs when using cat bonds, i.e., a static hedging approach is sufficient, which also reduces transaction costs and possible tracking errors. Therefore, the liquidity requirement is reduced to the availability of suitable cat bonds at the time of replication. We derive prices for binary / non-indemnity-based ILWs, where the payout only depends on the industry loss index exceeding a contractually defined trigger level

⁴ The prices of available index-linked catastrophic loss instruments such as ILWs should generally be consistent with the prices of other derivatives traded on an already liquid market such as in the case of cat bonds. To ensure this consistency, the prices of ILWs should equal the prices of replicating portfolios consisting of tradable derivatives (cat bonds). This is also generally in line with the findings in Jaeger, Müller and Scherling (2010).

during the contract term. If a suitable cat bond is not available for deriving ILW prices, we provide proper approximations under some additional assumptions. To illustrate and test the proposed approximations in case of index-linked instruments, we derive the price of an ILW using secondary market cat bond prices and compare the resulting price with the available real-world ILW prices, finding a high degree of consistency, which supports our suggested approximations for replicating portfolios. Moreover, as a further application, we approximate prices of cat bonds using empirical data and compare them with real secondary market data in order to examine whether the market prices consistently.

In the case of instruments that are only index-linked (i.e., *non-indemnity-based*), one major advantage is that no assumptions concerning the distribution of the underlying industry loss index are necessary. In the case of *indemnity-based* index-linked catastrophic loss instruments, whose payout in addition to the industry loss index also depends on the company-specific loss caused by a catastrophe, one major problem is the behavior of the company loss, which should be tradable in some weak sense when using risk-neutral valuation. Therefore, we further suggest and briefly discuss three potential approaches for solving this issue. Furthermore, we examine the adequacy of our assumptions and indicate in which cases they can be used.

The presented approaches for pricing index-linked catastrophic loss instruments are of high relevance today and especially for the future (both for practical as well as academic endeavors), when index-linked catastrophic loss hedging instruments will become even more widespread than today and when some markets for derivatives like the cat bond market are truly liquid. One main contribution of our work is to overcome the crucial point of the tradability of the loss index through suitable approximations and to provide explicit pricing formulas using replication techniques. While the focus of the paper is primarily theoretical, we use empirical examples to illustrate the suggested approaches by comparing ILW and cat bond prices with the ones derived based on the replication and approximation techniques.

The remainder of the paper is structured as follows. Section 2 gives an overview of related literature with a focus on the underlying theory and assumptions. Section 3 introduces index-linked catastrophic loss instruments and ILWs as representatives of index-linked catastrophic loss hedging instruments, their basic properties and the underlying industry loss indices. In Section 4 we present the pricing approaches and as an example apply the approaches to consistently price ILWs and cat bonds in Section 5. Section 6 gives an outlook on the pricing of indemnity-based products, and Section 7 concludes.

2. FURTHER RELATED LITERATURE

There are several papers which deal with the pricing of index-linked catastrophic loss instruments such as ILWs or strongly related products. Actuarial pricing principles are applied to ILWs in Gatzert and Schmeiser (2012), for instance, while Gründl and Schmeiser (2002) compare actuarial pricing approaches with the capital asset pricing model to calculate prices for double-trigger reinsurance contracts, which are similar to indemnity-based ILWs. Furthermore, there are papers that combine financial and actuarial pricing approaches such as Møller (2002, 2003), who discusses the valuation and hedging of insurance products that depend on both the financial market and insurance claims. Regarding the arbitrage-free pricing of ART instruments related to index-linked catastrophic loss instruments, there is a wide literature, which is outlined in the following.

First, Cummins and Geman (1995) develop a model to price cat futures and call spreads written on the aggregated claims process using an arbitrage approach based on a jump diffusion model. In contrast, Bakshi and Madan (2002) price options written on the average level of a Markov process using a mean-reverting process and derive closed-form solutions for cat option prices. Haslip and Kaishev (2010) price reinsurance contracts with specific focus on catastrophe losses. They assume a liquid market of indemnity-based cat bonds and that the aggregated loss process of a company follows a compound Poisson process, and then calculate arbitrage-free prices for the excess of loss reinsurance contracts using Fourier transformations. In contrast to their setting, we only assume a liquid cat bond market with comparable index-based cat bonds and weaken the distributional assumptions. Prices for catastrophe equity puts, which are double-trigger contracts that can only be exercised if the insured loss rises above a certain level, are derived in Jaimungal and Wang (2006), who extend the work of Cox, Fairchild and Pedersen (2004) by allowing non-constant interest rates and non-constant losses for every occurring catastrophe. The authors derive an explicit formula and the Greeks for the price of the option using standard arbitrage-free option pricing theory and based on the Merton (1976) approach. This assumes the insured loss follows a compound Poisson process, the stock price process follows a geometric Brownian motion between the jumps of the compound Poisson process, and that the influence of the insured loss on the stock price process is relative, i.e., the absolute loss of the stock price depends on its value before a jump happens.

Integral to one of our pricing approaches for ILWs is the valuation of cat bonds. In this context, Bantwal and Kunreuther (2000) observe that the spreads of cat bonds do not necessarily align with standard investor preferences and draw on behavioral economics as one explanation for this observation. Furthermore, due to jumps induced by natural catastrophes,

the market of cat bonds is generally incomplete. To deal with incompleteness, one approach is offered by Merton (1976), who argues that the risk of jumps can be completely neutralized by diversification, as jumps are of an idiosyncratic nature. Under this “risk-neutralized” assumption, one can employ arbitrage-free pricing.⁵ In Cox and Pedersen (2000) and Cox, Fairchild and Pedersen (2004), the pricing of catastrophe bonds has also been discussed, where the latter deal with the dynamics and interactions of losses and share value. In Poncet and Vaugirard (2002), a pricing approach is introduced that uses stochastic interest rates and a diffusion process for the industry loss index without jump risk from catastrophes, while in Vaugirard (2003a) jumps are accounted for by means of a jump diffusion process.⁶ The consistent pricing of double-trigger cat bonds based on single trigger bonds is studied in Lane (2004), who points out that in the absence of arbitrage, the sum of the prices of two single trigger cat bonds should generally equal the sum of the prices of two fitting double-trigger cat bonds (one junior and one senior tranche). In Section 5, we further develop this idea when applying our approaches to the consistent pricing of single trigger cat bonds and without assuming perfectly fitting cat bonds. An empirical example for a potential security-class arbitrage opportunity is discussed in Jaeger, Müller and Scherling (2010, p. 27f), who show that the same risk may be priced differently depending on the type of security using ILWs and cat bonds as an example.⁷ However, the authors also emphasize the different contract terms as one potential main reason for observed differences in prices. Based on their empirical data they further critically evaluate other types of arbitrage, stating that there may be rather small program replication arbitrage opportunities (mainly arising from other risks or transaction costs), and regarding trigger type arbitrage find no evidence in their historical data.

Apart from that, Galeotti, Guertler and Winkelvos (2012) empirically compare different pricing models for cat bonds using primary market data and find that the Wang (2000) transform (using the student-t-distribution) as well as linear models provide the most accurate fit if the financial crisis is not taken into account in the dataset. If the latter is included in the analysis, all models provide similar results. Also based on primary market data, Braun (2014)

⁵ See, e.g., Lee and Yu (2002) for an application of this approach to pricing cat bonds using risk-neutral valuation under default risk, basis risk, and moral hazard. Alternatively, an equilibrium model can be employed (see, e.g. Zhu (2011), and Cox and Pedersen (2000) for a brief discussion on the relation of the two frameworks).

⁶ Other pricing approaches are based on equilibrium models (see Dieckmann (2010)) or employ the pricing techniques of credit risk by means of probability-of-default and loss-given-default, which translate to probability-of-catastrophe and loss-given-catastrophe (see Jarrow (2010)). Moreover, Dieckmann (2010) shows that the term structure of cat bonds is slightly upwards sloping, which can partly be explained by the equilibrium assumption.

⁷ The example (referring to March 2007) suggested buying the (three-year) Mystic Re cat bond on Northeast U.S. wind with an industry loss trigger of \$30 to \$40 billion and linear payout with a spread of LIBOR + 700 bps and buying a (twelve months) binary ILW on Northeast U.S. wind with a \$30 billion trigger level for 615 bps (p. 27).

proposes an econometric pricing model for cat bonds and shows that the expected loss is the most relevant determinant of the cat bond spread at issuance, while territory, sponsor, reinsurance cycle, and spreads on comparably rated corporate bonds also have a considerable impact. Indemnity-triggers, in contrast, do not imply a higher spread for investors despite the risk of moral hazard, which the author explains with an increasing rate of acceptance among investors as well as incentive provisions, where sponsor and investor proportionally share the risk above a certain threshold. Guertler, Hibbeln and Winkelvos (2014) use secondary market data to study the impact of natural catastrophes and the financial crisis on cat bond prices and to identify factors that influence premiums, showing that especially ratings play a role, in contrast to indemnity-triggers, which is similar to Braun (2014).

3. MODELING INDEX-LINKED CATASTROPHIC LOSS INSTRUMENTS

One main property of index-linked instruments such as ILWs is that they depend on an industry loss index, which generally does not instantaneously reflect the exact insured damage. Instead, if a catastrophe occurs, the first value of the index is a preliminary estimation (see, e.g., Kerney (2013a)), which is adjusted if necessary or at predetermined points in time (see PCS (2013) and PERILS (2013)). To take this development into account, we follow the approach of Vaugirard (2003a, 2003b) and assume the actual (contractually defined) maturity T' of the index-linked catastrophic loss instruments to exceed the *risk exposure period* T , during which the catastrophe must occur in order to trigger a payoff. Newly occurring catastrophes between the end of the risk exposure period T and the maturity T' are not taken into account, but adjustments of the loss estimates from the catastrophes that occurred during the risk exposure period (between time 0 and T) are taken into account and are reflected in the index. Hence, while the time interval from 0 to T represents the *risk exposure period*, the time interval from T to T' is referred to as the *development period*. Thus, an industry loss index incorporates data from insured losses arising from an occurring catastrophe and each qualifying event is reflected in the index.

In the following, we consider an aggregated industry loss index I^T for the risk exposure period until time T by aggregating estimated losses from all catastrophes (e.g., of the same and contractually defined type) that occurred during the risk exposure period.⁸ Hence, following, e.g., Biagini, Bregman and Meyer-Brandis (2008), the aggregated industry loss index I^T for the risk exposure period until T , taking into account the development of the estimations until time $t \leq T'$, is given by

⁸ This loss information is provided by specialized industry loss index providers. Note that this also depends on the type of index-linked catastrophic loss instruments. E.g., if dealing with occurrence ILWs (see Ishaq (2005)), whose payoff does not depend on the sum of insured damage but only on the loss caused by the first event), one needs a different approach, but our techniques are applicable as well.

$$I_t^T = \sum_{i=1}^{N_{t \wedge T}} X_t^i,$$

where $N_{t \wedge T}$ (with $t \wedge T = \min(t, T)$) is the number of catastrophes that occurred up to time t within the risk exposure period until T and X_t^i denotes the time-dependent estimation of the insured loss arising due to the i -th catastrophe at time t . In the following, we focus on index-linked catastrophic loss instruments, whose payoffs depend on the industry loss index at maturity T' and which can be defined by

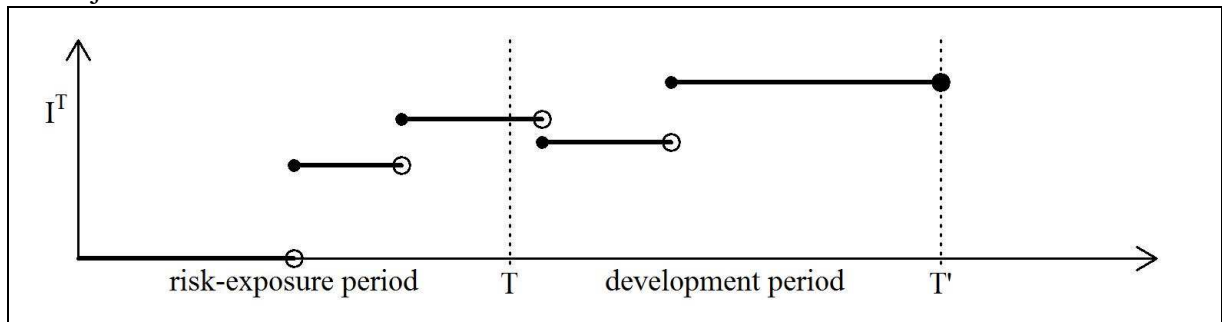
$$P_{T'}^T = h(I_{T'}^T),$$

where h is a non-negative measurable function. There are also *indemnity-based* index-linked catastrophic loss instruments, which additionally depend on the company loss. The payoff at time T' of such indemnity-based index-linked catastrophic loss instrument is analogously given by

$$P_{T'}^T = h(I_{T'}^T, L_{T'}^T), \quad (1)$$

where L^T denotes (analogously to I^T) the estimated aggregated company loss for the risk exposure period until T . In Figure 1, we depict the development of an industry loss index for a single catastrophe. After the catastrophe occurred, the loss estimations are adjusted several times, and additionally occurring catastrophes during the risk exposure period would cause additional jumps. The solid dot on the right side corresponds to the value of the industry loss index at maturity and hence the payoff is given by applying the function h to this value. Note that while downward adjustments of the estimation are possible in principle, they are unlikely and adjustments are typically upward.⁹

Figure 1: Exemplary development of an industry loss index in the case of one catastrophe and adjustments of loss estimations over time



⁹ See, e.g., McDonnell (2002) for the development of the Property Claim Service estimations of the ten U.S. natural catastrophes with the highest insured damage.

We assume that I^T is given by an industry loss index provider such as, e.g., the Property Claim Service (PCS) index for the U.S. or the PERILS index for Europe. It represents an estimation of the total insured loss caused by a catastrophe. PCS provides insured property loss and defines a catastrophe as an event that causes at least \$25 million in direct insured property losses and affects a significant number of policyholders and insurers. The PCS index covers perils including earthquake, fire, hail, hurricane, terrorism, utility service disruption and winter storm. The loss data is updated approximately every 60 days, if the event caused more than \$250 million in insured property losses, until PCS believes the estimated loss reflects the insured loss for the industry (see PCS (2013) and Kerney (2013a)). PERILS defines a catastrophe as an event exceeding a total loss of €200 million and estimates only property windstorm losses in Europe and property flood losses in the U.K. It publishes the first index value report six weeks after the event at the latest and updates after three, six and twelve months. More updates are provided if necessary but the reporting is final in any case after 36 months (see PERILS (2013)). It is important to keep in mind that these indices do not incorporate information instantaneously but with a certain delay; however under the assumption of sufficient liquidity in the derivatives market, we can assume that information is incorporated almost instantaneously (similar to the CDS market for credit defaults) into the derivatives prices (e.g., cat bonds that play a crucial role here).

One main problem related to the use of an index-linked catastrophic loss instrument in the context of hedging is basis risk, which arises if the dependence between the index and the company's losses (that are to be hedged) is not sufficiently high. In particular, the company loss could be high, but the industry loss index could be too low to trigger a payoff. There are several definitions of basis risk and we refer to Cummins, Lalonde and Phillips (2004) for a discussion of basis risk associated with index-linked catastrophic loss instruments.

4. PRICING INDEX-LINKED CATASTROPHIC LOSS INSTRUMENTS

4.1 Pricing by replication under the no-arbitrage assumption

In general, actuarial valuation methods for insurance contracts are based on the expected loss plus a specific loading, which depends, e.g., on the insurers' risk aversion and its already existing portfolio (see Gatzert and Schmeiser, 2012). Financial pricing approaches generally allow the derivation of market-consistent prices. For arbitrage-free valuation, one has to calculate the expected value under a certain risk-neutral measure Q given the existence of a liquid market and independent of the existing portfolio. If there is a unique risk-neutral measure (i.e., the market is complete) the arbitrage-free price is unique and the price corresponds to the initial value of a self-financing portfolio replicating the cash flow (Harrison and Kreps, 1979).

In the present setting, we focus on the two stochastic processes of importance: the industry loss index I^T and the company loss L^T . These processes need to be tradable in a more general sense, as there is no possibility to trade I^T or L^T directly like a stock and it is unrealistic to assume that there is a possibility to buy or sell this index; however, we assume there is a liquid market for certain derivatives on the industry loss index I^T .¹⁰ To avoid arbitrage opportunities, the prices of these derivatives should coincide with the expectation of the discounted cash flow under a risk-neutral measure (unique or not, depending on the market's completeness), and the prices of other index-linked catastrophic loss instruments should be consistent with the prices of the derivatives already traded on a liquid market (given a sufficient degree of comparability in regard to transaction costs, for instance). To ensure this consistency, the prices of these other instruments should equal the expectations under the same risk-neutral measure, or, equivalently, should equal the prices of replicating portfolios consisting of tradable derivatives, for instance. In case the market is incomplete, i.e., there are not sufficient traded derivatives available (no exact replicating portfolio; only partial hedging is possible), other approaches can be used, such as sub-/super-hedging (e.g., Bertsimas, Kogan and Lo, 2001), by choosing a self-financing portfolio based on maximizing expected utility (e.g., Henderson, 2002), or by selecting a risk-neutral measure according to certain criteria (e.g., jump risk is not priced, see Merton (1976), Delbaen and Schachermayer (1996)).

We thus assume that the liquid market for certain index-linked catastrophic loss instruments (e.g., cat bonds) is at least large enough to allow the derivation of an exact replicating portfolio or at least a close approximation. For instance, in case suitable cat bonds are available for perfectly replicating the respective instrument's cash flows (e.g., an ILW or another cat bond), a unique arbitrage-free price can be derived.¹¹ Alternatively, in case the required cat bonds are not available (e.g., mismatching trigger level or time to maturity), we derive suitable approximations for the replicating portfolio. Using direct replication also has the significant advantage that a risk-neutral measure does not have to be specified and that model risk is significantly reduced for the hedger.

¹⁰ These derivatives also represent index-linked catastrophic loss instruments. Note, however, that the assumption of liquidity only holds for certain types of derivatives, as standard call and put options on an industry loss index are not traded on a liquid market. Call and put options were introduced 1995 by CBOT, but due to limited trading these options were delisted in 2000 (see, e.g., Cummins and Weiss (2009)). Hence, the derivatives we focus on in the following are cat bonds, which exhibit a considerable market volume with a relevant secondary market. In case that there is a liquid derivatives market in the sense that calls and puts on the underlying industry loss index are traded at all strikes, the Litzenberg formula could be applied to for static replication.

¹¹ The instruments we focus on are not path-dependent and there could be various measures resulting in the same prices for instruments with a payoff depending only on $I_{T'}^T$. Since there is no difference for the prices of the instruments we consider, it is not relevant which of these measures we take.

In case of *indemnity-based* index-linked catastrophic loss instruments, one crucial point and major problem in addition to the treatment of the industry loss index is the tradability of the company loss L^T . As potential remedies we discuss three preliminary approaches with varying degrees of assumptions in Section 6.

4.2 Industry Loss Warranties

We apply the proposed approaches to price binary ILWs, whose payoff depends on whether the industry loss index exceeds the trigger level Y at the end of the development period T' (see McDonnell (2002)). The payoff at maturity T' of a *binary ILW* (denoted “ b ”) with trigger level Y and risk exposure period until T is given by

$$P_{ILW,T'}^{b,Y,T} = D \cdot \mathbf{1}_{\{I_{T'}^T > Y\}}, \quad (2)$$

where D represents the possible payout and $\mathbf{1}_{\{I_{T'}^T > Y\}}$ denotes the indicator function, which is equal to 1 if $I_{T'}^T > Y$, i.e., if the industry loss index at maturity $I_{T'}^T$ exceeds the trigger level Y (due to catastrophes that occurred during the risk exposure time T), and 0 otherwise.¹²

To illustrate our approach more specifically, we assume that there is a liquid cat bond market consisting of cat bonds (with comparable maturities and strikes etc.) that can be used to derive consistent ILW prices by means of replicating the ILWs' cash flows.¹³ This extends the approach of Haslip and Kaishev (2010), who also assume a liquid cat bond market but – in contrast to our analysis – focus on indemnity-based cat bonds, and then evaluate excess of loss reinsurance contracts under additional assumptions concerning the risk-neutral measure Q . The assumption of a liquid cat bond market can be considered reasonably realistic given the high volume of cat bonds (more than \$17 billion of capital outstanding in 2013 (see AON (2013)) and a growing secondary market with a considerable volume of cat bond transactions (see Moody (2013)). Furthermore, sponsors are becoming more and more familiar with this kind of risk transfer and service providers involved in structuring and marketing exhibit an increasing experience. In addition, frictional costs of cat bond transactions are decreasing

¹² Note that an alternative but less common representation of an ILW would be that the payoff is triggered if the index exceeds the trigger limit during the contract term (knock-in barrier option). The following analysis can be extended to this case as well.

¹³ In general, in case the market is incomplete and ILWs cannot be fully replicated, under suitable assumptions it can still be possible to at least approximately (dynamically) replicate the ILWs (see, e.g., Bertsimas, Kogan and Lo (2001), Xu (2006), where arbitrage-free price windows are derived by means of sub- and super-hedges). With the derivation of sub- and super-hedges, one obtains a price interval in which any arbitrage-free price of the ILW has to be contained. Within the interval, the chosen price will depend on the risk preference of the investor and might still be subject to inefficiencies.

(less than 20 basis points, see Kerney (2013b)). Time to market and frictional costs are further improved by the use of different securitization structures involving a Special Purpose Insurer (SPI) instead of a Special Purpose Vehicle (SPV), which can be set up in less than three to four weeks overall (see Garrod (2014)), thus also improving the degree of liquidity in the cat bond market. In addition, Braun (2014) points out that sponsors recently started to increasingly use shelf-offerings (e.g. Swiss Re Successor Series), where additional classes of notes can be issued repeatedly out of the same SPV, which eases access to capacity and considerably reduces transaction costs. These developments also contribute to reducing differences in transaction costs and transaction times between cat bonds and ILWs, as the latter are generally more easily available and less costly.¹⁴ However, we show that liquidity is not needed to the same extent as in classical option pricing as continuous replication is not necessary to replicate ILWs when using cat bonds, i.e., a static hedging approach is sufficient. Moreover, the Bermuda Stock Exchange, among others (see Artemis (2013)), has started to list cat bonds, which is a first step to trading them on the exchange, indicating that the secondary market is growing, the transaction time on the primary market and the frictional costs are decreasing and even if the market is not yet enough liquid, it appears reasonable to assume that it will become liquid in the foreseeable future in view of the current developments.

The approach using available cat bonds in order to (at least approximately) replicate cash flows can also be used for pricing other index-linked catastrophic loss instruments covering a wide variety of instruments. Hence, once the replicating portfolio is known, prices can easily be calculated as they can directly be observed in the market. In addition, this approach can be used to test the degree of liquidity in the market. Consistent prices would encourage the assumption of a liquid market, while inconsistent prices would contradict this assumption and lead to an arbitrage opportunity or indicate the existence of an external risk that might not have been incorporated into market prices.

4.3 Pricing binary ILWs by replication using cat bonds

In the following, we present different ways of replicating cash flows of binary ILWs. In case of a liquid cat bond market, the whole market is represented by the filtered probability space $(\Omega, \mathcal{F}, (F_t)_{t \geq 0}, P)$, where $(F_t)_{t \geq 0}$ is a filtration satisfying the usual assumptions and F_t represents all information up to time t . Following the fundamental theorem of asset pricing (see, e.g., Delbaen and Schachermayer (1994)), the price of every contingent claim is given by the expectation of the discounted payoff under an equivalent martingale measure Q .

¹⁴ Note that in case the approaches presented here are applied to consistently price cat bonds, transaction costs would be even more comparable.

The discounted payoff (at time 0) of an index-based cat bond with binary payoff (denoted “ b ”),¹⁵ trigger level Y , risk exposure period until T , coupon payment c and maturing at time T' and without loss of generality an assumed nominal of 1 is generally given by

$$P_{cat,T'}^{b,Y,T} = \sum_{j=1}^n c \cdot e^{-r \cdot t_j} \mathbf{1}_{\{I_{t_j}^T \leq Y\}} + e^{-r \cdot T'} \mathbf{1}_{\{I_{T'}^T \leq Y\}}, \quad (3)$$

where r is the constant risk-free interest rate (this assumption can be weakened) and n is the number of coupon payments (coupon time intervals). In the following, we first assume that there is a perfectly fitting cat bond, i.e., a zero coupon ($c = 0$) cat bond with the same trigger level Y , the same risk exposure period until T and maturing at the same time T' , and then extend our formula to imperfectly fitting cat bonds under certain additional assumptions.

Matching maturity and trigger level

First, if there are binary zero coupon cat bonds with the same maturity and the same trigger level as the ILW, we can proceed as follows. According to Equation (3), the risk-neutral price at time t of a zero coupon (denoted “0”) cat bond with binary payoff, trigger level Y , risk exposure period until T , maturing at time T' is (see Haslip and Kaishev (2010))

$$V_{cat,T'}^{b,0,Y,T}(t) = E^Q \left(e^{-r(T'-t)} \mathbf{1}_{\{I_t^T \leq Y\}} / F_t \right) = e^{-r(T'-t)} - E^Q \left(e^{-r(T'-t)} \mathbf{1}_{\{I_t^T > Y\}} / F_t \right). \quad (4)$$

Hence, together with (2) (and $D = 1$), Equation (4) turns into

$$V_{cat,T'}^{b,0,Y,T}(t) = e^{-r(T'-t)} - V_{ILW,T'}^{b,Y,T}(t),$$

where $V_{ILW,T'}^{b,Y,T}$ is the price of a binary ILW with risk exposure period until T , trigger level Y and maturing at time T' .¹⁶ Thus, in the presence of a zero coupon cat bond with the same trigger level Y and same maturity, the price is given by

$$V_{ILW,T'}^{b,Y,T}(t) = E^Q \left(e^{-r(T'-t)} \mathbf{1}_{\{I_t^T > Y\}} / F_t \right) = e^{-r(T'-t)} - V_{cat,T'}^{b,0,Y,T}(t), \quad (5)$$

¹⁵ Even though proportional payouts are generally more common in practice, the Mexican cat bond issued 2009, for instance, featured a binary trigger (see Cummins, 2008).

¹⁶ A related observation is made by Vaugirard (2003a), who observes that the buyer of a catastrophe bond holds a short position on a binary option on the index.

where $e^{-r(T'-t)}$ is the price of the related zero coupon bond without catastrophe (or any other default) risk under the assumption of a constant interest rate. In case of a non-constant interest rate this could be replaced with the price of a zero coupon bond maturing at time T' without default risk. For the remainder of the paper, we assume without loss of generality that $t = 0$. Note that since direct replication is used, independence between the risk-free rate and the industry loss index is not necessary in this case. Equation (5) thus also yields a replicating portfolio for a binary ILW. In particular, the cash flow of a binary ILW with risk exposure period until T , trigger level Y maturing at time T' can be perfectly replicated by buying a zero coupon bond without default risk and selling a binary zero coupon cat bond with risk exposure period until T , trigger level Y and maturing at time T' .

One has to take into account that the chance of finding a perfectly fitting cat bond in the market might be low, such that an approximation in spite of a maturity mismatch might be easier. Therefore, we further provide approximations if there are only cat bonds with different maturity, different trigger level, non-zero or non-binary coupons available.

Matching trigger level but mismatching maturity

We next assume that there is a liquid market for binary zero coupon cat bonds with the same trigger level Y but different maturities and different risk exposure periods, thus implying an incomplete market setting. In this case, sub- and super-hedging can be applied, where the arbitrage-free price of an ILW with maturity T' lies between the prices of ILWs with shorter and longer time to maturity, both given by Equation (5), which results in an upper and lower bound for the ILW price. The obtained range can be rather large, which is why we choose the zero coupon cat bond with the maturity \tilde{T}' and risk exposure period until \tilde{T} that is closest to the parameters of the ILW we intend to price (everything else equal, i.e., trigger level and index). The aim is thus to approximate the binary ILW price $V_{ILW,T'}^{b,Y,T}$ with the price of this cat bond, which can be observed in the market. First, the price of a binary ILW with risk exposure period until \tilde{T} , trigger level Y and maturing at time \tilde{T}' is given through Equation (5) by

$$V_{ILW,\tilde{T}'}^{b,Y,\tilde{T}}(0) = e^{-r\tilde{T}'} E^Q \left(\mathbf{1}_{\{I_{\tilde{T}'}^{\tilde{T}} > Y\}} \right) = e^{-r\tilde{T}'} - V_{cat,\tilde{T}'}^{b,0,Y,\tilde{T}}(0), \quad (6)$$

which can then be used to price the ILW with different maturity and different risk exposure period by

$$\begin{aligned}
V_{ILW,T'}^{b,Y,T}(0) &= e^{-rT'} E^Q \left(1_{\{I_{T'}^T > Y\}} \right) = e^{-rT'} E^Q \left(1_{\{I_{\tilde{T}'}^{\tilde{T}} > Y\}} + \left(1_{\{I_{T'}^T > Y\}} - 1_{\{I_{\tilde{T}'}^{\tilde{T}} > Y\}} \right) \right) \\
&= e^{-rT'} e^{r\tilde{T}'} V_{ILW,\tilde{T}'}^{b,Y,\tilde{T}}(0) + D_{\tilde{T},\tilde{T}'}^{T,T'} = e^{-rT'} e^{r\tilde{T}'} \left(e^{-r\tilde{T}'} - V_{cat,\tilde{T}'}^{b,0,Y,\tilde{T}}(0) \right) + D_{\tilde{T},\tilde{T}'}^{T,T'} \\
&= e^{-rT'} - e^{-r(T'-\tilde{T}')} V_{cat,\tilde{T}'}^{b,0,Y,\tilde{T}}(0) + D_{\tilde{T},\tilde{T}'}^{T,T'},
\end{aligned} \tag{7}$$

where

$$D_{\tilde{T},\tilde{T}'}^{T,T'} = e^{-rT'} E^Q \left(1_{\{I_{T'}^T > Y\}} - 1_{\{I_{\tilde{T}'}^{\tilde{T}} > Y\}} \right)$$

is the residual difference in (7) arising from the approximation error due to using an observable traded cat bond with mismatching maturity (closest to the one of the ILW). To calculate $D_{\tilde{T},\tilde{T}'}^{T,T'}$, additional assumptions regarding the distribution of I are needed. However, if the maturities are close, the approximation difference $D_{\tilde{T},\tilde{T}'}^{T,T'}$ is small and would only marginally depend on the underlying distribution assumptions.

To exemplarily calculate $D_{\tilde{T},\tilde{T}'}^{T,T'}$ for a realistic scenario, let the risk exposure period be equal to the maturity ($T = T'$, $\tilde{T} = \tilde{T}'$) and assume that the industry loss index reflects the real catastrophe losses instantaneously. Furthermore, let I^T follow a compound Poisson process under Q , which is not too restrictive if I^T follows a compound Poisson process under the real world measure P . Delbaen and Haezendonck (1989) showed under some reasonable assumptions, which should be fulfilled in most non-life insurance cases, that I^T remains a compound Poisson process under any equivalent risk-neutral measure. For a further treatment, we refer to Mürmann (2008) and Aase (1992). These assumptions lead to

$$I_{\tilde{T}'}^{\tilde{T}} = \sum_{i=1}^{N_{\tilde{T}'}} X^i, \tag{8}$$

where $N_{\tilde{T}'}$ and X^i are independent, $N_{\tilde{T}'}$ denotes the claims arrival process (i.e., a Poisson process with intensity λ) and X^i the i.i.d. claims (see, e.g. Levi and Partrat (1991)). Furthermore, we assume that the industry loss index incorporates the actual catastrophe losses instantaneously and if two or more catastrophes occur, the industry loss index almost surely exceeds the trigger level.

Using these assumptions and a resulting $D_{\tilde{T},\tilde{T}'}^{T,T'}$ as shown in the Appendix, we can approximate the ILW price through

$$V_{ILW,T}^{b,Y,T}(0) = e^{-rT} + e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda(T-\tilde{T})} \frac{T}{\tilde{T}} \left(e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda\tilde{T}} \right) \right). \quad (9)$$

Note that combining market data (available cat bond price) and model assumptions regarding the difference term D in (7) will generally reduce potential model risk involved in pricing index-linked catastrophic instruments, which would be considerably higher when applying the distributional assumptions to price the ILW without using any market data.

Matching maturity but mismatching trigger level

If the maturity of the cat bond and the ILW is the same, but the trigger level Y differs, bounds for prices can be obtained by analogously applying the approach in the last section (Equation (7)). We similarly obtain

$$\begin{aligned} V_{ILW,T'}^{b,Y,T}(0) &= e^{-rT'} E^Q \left(\mathbf{1}_{\{I_{T'}^T > Y\}} \right) = e^{-rT'} E^Q \left(\mathbf{1}_{\{I_{T'}^T > \tilde{Y}\}} + \left(\mathbf{1}_{\{I_{T'}^T > Y\}} - \mathbf{1}_{\{I_{T'}^T > \tilde{Y}\}} \right) \right) \\ &= V_{ILW,T'}^{b,\tilde{Y},T}(0) + D_{\tilde{Y}}^Y = e^{-rT'} - V_{cat,T'}^{b,0,\tilde{Y},T}(0) + D_{\tilde{Y}}^Y \end{aligned} \quad (10)$$

for the price of a binary ILW with trigger level Y , risk exposure period until T and maturing at time T' with an approximation error (residual difference)

$$D_{\tilde{Y}}^Y = e^{-rT'} E^Q \left(\mathbf{1}_{\{I_{T'}^T > Y\}} - \mathbf{1}_{\{I_{T'}^T > \tilde{Y}\}} \right).$$

Using the same assumptions as in the previous subsection and following Burnecki, Kukla and Weron (2000) and Katz (2002), who showed that the log-normal distribution provides the best fit to the PCS index among typical claim size distributions¹⁷, we assume that the i.i.d. claims X^i follow a log-normal distribution with parameter μ and σ^2 . In this case we can derive an approximation for $Q(I_{T'}^T > Y)$ using the assumptions from the last subsection (see Appendix A), i.e.,

$$\begin{aligned} Q(I_{T'}^T > \tilde{Y}) &\approx 1 - e^{-\lambda T'} - e^{-\lambda T'} \lambda T' Q(X^1 \leq \tilde{Y}) \\ &= 1 - e^{-\lambda T'} - e^{-\lambda T'} \lambda T' \Phi \left(\frac{\ln(\tilde{Y}) - \mu}{\sigma} \right), \end{aligned} \quad (11)$$

¹⁷ Note that the distribution under the risk-neutral measure Q can be different, but when using the Wang transform (see Wang (2000)), for instance, one obtains a log-normal distribution under Q , too, but with adjusted parameter.

where Φ is the standard normal distribution function. Equation (11) then leads to

$$\begin{aligned} D_{\tilde{Y}}^Y &= e^{-rT'} E^Q \left(1_{\{I_{T'}^r > Y\}} - 1_{\{I_{T'}^r > \tilde{Y}\}} \right) \\ &= e^{-rT'} \left(1 - e^{-\lambda T'} - e^{-\lambda T'} \lambda T' \Phi \left(\frac{\ln(Y) - \mu}{\sigma} \right) \right) - e^{-rT'} \left(1 - e^{-\lambda T'} - e^{-\lambda T'} \lambda T' \Phi \left(\frac{\ln(\tilde{Y}) - \mu}{\sigma} \right) \right) \\ &= e^{-rT'} e^{-\lambda T'} \lambda T' \left(-\Phi \left(\frac{\ln(Y) - \mu}{\sigma} \right) + \Phi \left(\frac{\ln(\tilde{Y}) - \mu}{\sigma} \right) \right) \end{aligned}$$

which can be used to solve Equation (10) under our simplifying assumptions.

Non-zero coupon cat bonds

If there is no market for zero coupon cat bonds (as is currently the case), non-zero coupon cat bonds need to be used for approximating ILW prices. The decomposition follows the standard bond stripping approach, where each coupon itself is understood as a zero coupon bond with adjusted nominal, interest rate and maturity. To calculate the price of an ILW involving only the non-zero coupon cat bond price $V_{cat,T'}^{b,Y,T}$, we use Equation (6). Recall Equation (3) for the payoff of such a cat bond and Equation (7) for adjusting a mismatching maturity to obtain

$$\begin{aligned} V_{cat,T'}^{b,Y,T}(0) &= \sum_{j=1}^n c \cdot E^Q \left(e^{-rt_j} 1_{\{I_{t_j}^r \leq Y\}} \right) + E^Q \left(e^{-rT'} 1_{\{I_{T'}^r \leq Y\}} \right) \\ &= \sum_{j=1}^n c \cdot e^{-rt_j} E^Q \left(1 - 1_{\{I_{t_j}^r > Y\}} \right) + e^{-rT'} E^Q \left(1 - 1_{\{I_{T'}^r > Y\}} \right) \\ &= \sum_{j=1}^n c \cdot e^{-rt_j} - c \cdot \sum_{j=1}^n V_{ILW,t_j}^{b,Y,T}(0) + e^{-rT'} - V_{ILW,T'}^{b,Y,T}(0) \\ &= \sum_{j=1}^n c \cdot e^{-rt_j} - c \cdot \sum_{j=1}^n \left(e^{-r(T'-t_j)} V_{ILW,T'}^{b,Y,T}(0) + D_{T,T'}^{T,t_j} \right) + e^{-rT'} - V_{ILW,T'}^{b,Y,T}(0) \end{aligned}$$

Solving this equation for $V_{ILW,T'}^{b,Y,T}(0)$ leads to the price of a binary ILW with trigger level Y , risk exposure period until T and maturity T' , i.e.

$$V_{ILW,T'}^{b,Y,T}(0) = \left(e^{-rT'} - V_{cat,T'}^{b,Y,T}(0) + \sum_{j=1}^n c \cdot e^{-rt_j} - c \cdot \sum_{j=1}^n D_{T,T'}^{T,t_j} \right) \left(c \cdot \sum_{j=1}^n e^{-r(T'-t_j)} + 1 \right)^{-1}. \quad (12)$$

Cat bonds with non-binary payoff

We now assume a zero coupon cat bond with non-binary payoff, risk exposure period until T , trigger level Y , limit M , and maturing at time T' to have nominal 1 and that the payment at time T' depends on the extent (captured by $I_{T'}^T$) to which the industry loss index exceeds the trigger Y at time T' . The price of this cat bond is given by

$$\begin{aligned}
 V_{cat,T'}^{0,Y,M,T}(0) &= E^Q \left(e^{-rT'} \left(1 - \frac{\min\left((I_{T'}^T - Y)_+, M\right)}{M} \right) \right) \\
 &= e^{-rT'} - \frac{1}{M} \left(C_{T'}^{I^T,Y}(0) - C_{T'}^{I^T,Y+M}(0) \right) \\
 &= e^{-rT'} - \frac{1}{M} C_{T'}^{I^T,Y,Y+M}(0),
 \end{aligned} \tag{13}$$

where $C_{T'}^{I^T,Y}(0)$ is the price of a call option on the industry loss index I^T with strike price Y maturing at time T' and $C_{T'}^{I^T,Y,Y+M}(0)$ is the price of the associated call spread. We use the midpoint rectangle rule to approximate $C_{T'}^{I^T,Y,Y+M}(0)$ by

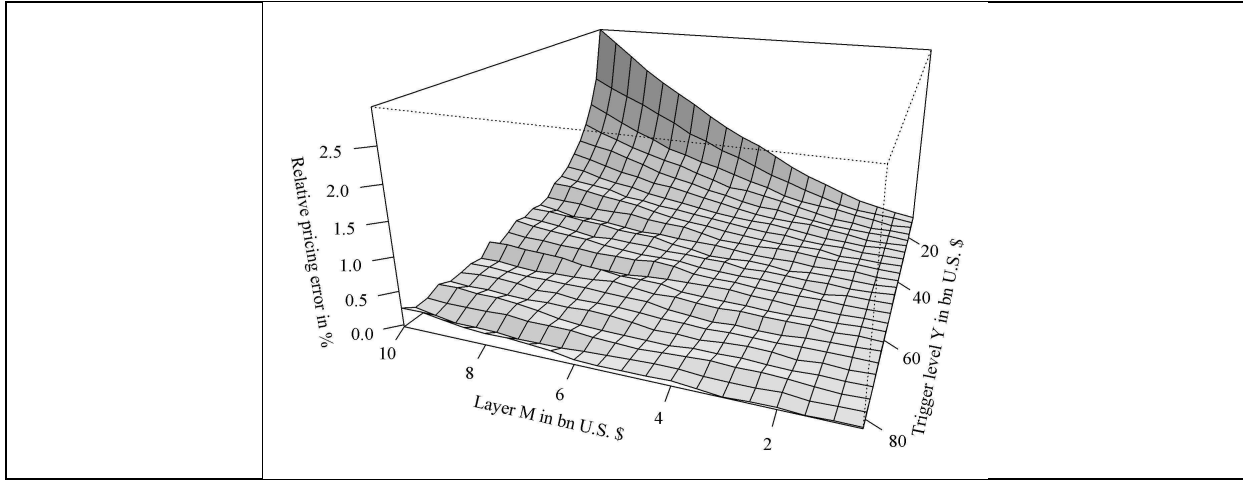
$$\begin{aligned}
 C_{T'}^{I^T,Y,Y+M}(0) &= E^Q \left(e^{-rT'} \min\left((I_{T'}^T - Y)_+, M\right) \right) \\
 &= \int_Y^{Y+M} e^{-rT'} Q(I_{T'}^T > x) dx = \int_Y^{Y+M} V_{ILW,T'}^{b,x,T}(0) dx \\
 &\approx M \cdot e^{-rT'} Q\left(I_{T'}^T > Y + \frac{M}{2}\right) = M \cdot V_{ILW,T'}^{b,Y+\frac{M}{2},T}(0)
 \end{aligned} \tag{14}$$

and use Equations (13) and (14) to obtain

$$V_{ILW,T'}^{b,Y+\frac{M}{2},T}(0) = e^{-rT'} - V_{cat,T'}^{b,0,Y+\frac{M}{2},T}(0) \approx e^{-rT'} - V_{cat,T'}^{0,Y,M,T}(0) \tag{15}$$

for the price of a binary ILW with trigger level $Y + \frac{1}{2}M$, risk exposure period until T and maturing at time T' . The approximation generally works well for small layers. Figure 2 illustrates the relative pricing error assuming compound Poisson losses with parameters from Katz (2002), which were estimated for individual North Atlantic tropical cyclones (primarily hurricanes) making landfall in the U.S. during the period 1925 to 1995.

Figure 2: Exemplary relative pricing error in percent through the approximation in Equation (15) (layer limit M and trigger level Y in bn. U.S. \$)



Notes: Monte-Carlo simulation based on 500,000 sample paths; layer and trigger level in bn U.S. \$. Relative pricing error in Equation (15) = $\left(c_T^{I,T} \cdot V_{ILW,T}^{b,Y+M}(0) \cdot M^{-1} - V_{ILW,T}^{b,Y+M,T}(0) \right) \cdot \left(V_{ILW,T}^{b,Y+M,T}(0) \right)^{-1}$. Parameters of the log-normal distribution according to Katz (2002): $\mu = -1.090$, $\sigma = 2.307$, the intensity of the compound Poisson process is $\lambda = 1.8$, maturity is set to 1.

Summary: Pricing binary ILWs

We presented different approaches for the valuation of binary ILWs under the assumption of no-arbitrage. In the presence of a liquid cat bond market, we obtain Equation (5) under no further assumptions. Since there may not be enough variety of cat bonds to match the parameters of every ILW, we further provided approximations for different trigger levels and maturities. To compute prices for ILWs (and similarly for other index-linked catastrophic loss instruments), the following three steps should be taken to obtain $V_{ILW,T}^{b,Y,T}$:

1. From non-zero cat bonds to zero coupon cat bonds.
2. Adjust the trigger level.
3. Adjust the maturity.

The possible cases, steps, and resulting approximations are summarized in Table 1.

Table 1: Overview of steps and solutions in the occurring cases for approximately calculating the prices of binary ILWs by means of replications using cat bonds

Cases	Steps and solutions	Equations
Zero coupon cat bond, matching maturity, matching trigger level	$V_{ILW,T'}^{b,Y,T}(0) = e^{-rT'} - V_{cat,T'}^{b,0,Y}(0)$	(5)
Zero coupon cat bond, mismatching maturity, matching trigger level	$V_{ILW,T'}^{b,Y,T}(t) = e^{-r(T'-\tilde{T}')}V_{ILW,\tilde{T}'}^{b,Y,\tilde{T}}(t) + D_{\tilde{T},T'}^{T,T'}$ with $V_{ILW,\tilde{T}'}^{b,Y,\tilde{T}}(0) = e^{-r\tilde{T}'} - V_{cat,\tilde{T}'}^{b,0,Y,\tilde{T}}(0)$	(6), (7)
Zero coupon cat bond, matching maturity, mismatching trigger level	$V_{ILW,T'}^{b,Y,T}(0) = V_{ILW,T'}^{b,\tilde{Y},T}(0) + D_{\tilde{Y}}^Y$ $V_{ILW,T'}^{b,\tilde{Y},T}(0) = e^{-rT'} - V_{cat,T'}^{b,0,\tilde{Y},T}(0)$	(10)
Zero coupon cat bond, mismatching maturity, mismatching trigger level	Eq. (10) to obtain $V_{ILW,T'}^{b,Y,T}(0)$ with $V_{ILW,T'}^{b,\tilde{Y},T}(0) = e^{-r(T'-\tilde{T}')}V_{ILW,\tilde{T}'}^{b,\tilde{Y},\tilde{T}}(t) + D_{\tilde{T},T'}^{T,T'}$ $V_{ILW,\tilde{T}'}^{b,\tilde{Y},\tilde{T}}(0) = e^{-r\tilde{T}'} - V_{cat,\tilde{T}'}^{b,0,\tilde{Y},\tilde{T}}(0)$	(6), (7), (10)
Non-zero coupon cat bond	Use Eq. (12) to obtain $V_{ILW,T'}^{b,Y,T}(0)$. Adjust trigger level and maturity if necessary	(6), (7), (10), (12)
Cat bonds with non-binary payoff	Use Eq. (15) to obtain $V_{ILW,T'}^{b,Y+0,5M,T}$. Adjust trigger level and maturity if necessary	(6), (7), (10), (15)

5. EMPIRICAL EXAMPLES FOR PRICING INDEX-LINKED CATASTROPHIC LOSS INSTRUMENTS BY REPLICATION USING CAT BONDS

5.1 Consistent pricing of binary ILWs using cat bonds

To illustrate and test the previously developed approximations, we use secondary market cat bond prices provided by Lane (2002) and compare the theoretically derived ILW prices obtained by means of our approximations (see Table 1) with the actual ILW prices dated from 04/01/2002 as provided by McDonnell (2002).¹⁸ To approximate the price of a binary ILW related to earthquake risk in California with a maturity of one year, the secondary market price of the cat bond “Western Capital”¹⁹ is used, which features the same underlying risk, but

¹⁸ Note that we need information regarding the attachment point and the layer of the cat bond, which in case of secondary (and primary) market data is typically not provided. Also, ILW prices are very difficult to obtain due to OTC transactions, which is why we use an empirical example from 2002 where these information were available.

¹⁹ Date of issue: 02/2001; maturity: 01/2003; coupon: 510; attachment point: \$22.5bn; exhaustion point: \$31.5bn; probability of first loss: 0.0082; probability of exhaustion: 0.0034; expected loss: 0.0055; index: PCS; risk: California earthquake (see Lane and Beckwith (2001), and Michel-Kerjan et al. (2011) for the attachment and exhaustion point). As we have no information regarding the risk exposure period, we assume that it equals the maturity.

mismatching maturity (10 months instead of one year) and a non-binary payoff (layer between \$22.5bn and \$31.5bn). The secondary market price is given as the spread (spreads published by Goldman Sachs at 03/31/2002: bid: 642 bp, ask: 554 bp (Lane, 2002)) over LIBOR (2.03% at²⁰ 03/31/2002). We use the developed approximations to derive a bid-ask interval for ILW prices and in the following exhibit the calculation for the ask price. Regarding the intensity of occurrence, due to a lack of alternative information we assume $\lambda = 0.0082$, which corresponds to the probability of the first loss,²¹ and define 03/31/2002 as $t = 0$. Since our approximations use zero coupon cat bond prices, we transform the secondary market spread using the LIBOR rate to a non-binary zero coupon price through

$$V_{cat,10/12}^{0,Y=22.5,M=31.5-22.5=9,10/12}(0) = \frac{1}{(1+0.0554+0.0203)^{\frac{10}{12}}} = 0.9410,$$

where 10 months (from 03/31/2002 to 01/31/2003) are left until the cat bond matures. According to Equation (15), we can use the midpoint rectangle rule to approximate the price of a zero coupon cat bond with binary payoff and trigger level $\frac{1}{2}(22.5+31.5) = 27$ (middle of the layer of the non-binary cat bond) by

$$V_{cat,T'=10/12}^{b,0,Y=\frac{22.5+31.5}{2}=27,T=10/12}(0) \approx V_{cat,T'=10/12}^{0,Y=22.5,M=31.5-22.5,T=10/12}(0) = 0.9410,$$

which, as illustrated in the previous section for the case of hurricanes, should generally work well for smaller layers and higher trigger level.²² The maturity of the ILWs, whose prices are stated in McDonnell (2002), is one year. For the adjustment of the mismatching maturity (from 10 months to one year) we use Equation (9) with $\lambda = 0.0082$, and the price of a binary ILW with trigger level of \$27bn is then derived by

$$V_{ILW,12/12}^{b,27,12/12}(0) = e^{-r \cdot 1} + e^{-r \cdot 1} \left(-e^{-\lambda \cdot 1} - e^{-\lambda(1-\frac{10}{12})} \frac{1}{\frac{10}{12}} \left(e^{\frac{r \cdot 10}{12}} \cdot V_{cat,10/12}^{b,0,27,10/12}(0) - e^{-\lambda \frac{10}{12}} \right) \right) \approx 5.06\%.$$

The ask price is thus given by 5.06% and the same calculation for the bid price yields 5.82%.²³ The actual prices stated by McDonnell (2002) are 5.25% for an ILW with a trigger

²⁰ See <http://de.global-rates.com>.

²¹ The probability of the first loss is the probability that the attachment point is exceeded (see, e.g., Galeotti, Gürtler and Winkler (2012, p. 405) for a formal description).

²² Taking Figure 2 (North Atlantic hurricane data instead of California earthquake) as a rough indication for the pricing error, the latter would amount to around 2-2.5%.

²³ We additionally calculated the outcome including coupon payments with resulting ask and bid prices of 5.09% and 5.85%.

level of \$25bn and 4.25% for trigger level of \$30bn, which already shows a high degree of consistency with our approximation when using the ask price (despite the use of an approximated intensity using the probability of first loss and the use of only partial information for the ILW price, for instance), whereas the difference is higher when using the bid price.

5.2 Consistent pricing of cat bond

Another application of our proposed approximation formulas is the consistent pricing of cat bonds. The available characteristics and prices of four selected cat bonds written on U.S. hurricane risk using the PCS industry loss index are summarized in Table 2.

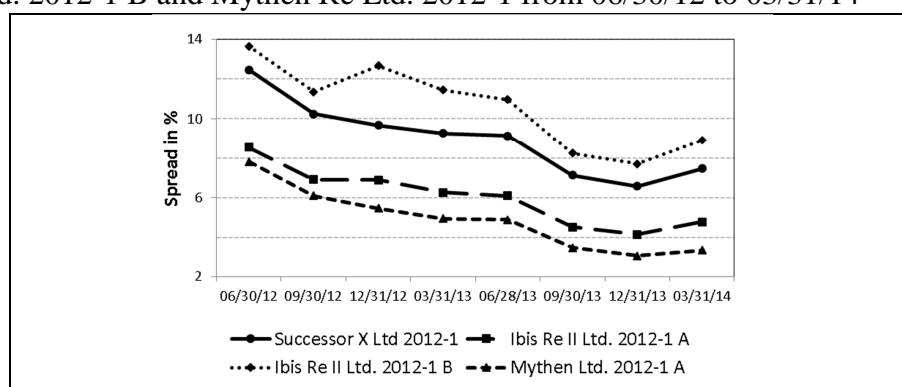
Table 2: Characteristics of Successor X Ltd 2012-1; Ibis Re II Ltd. 2012-1 A; Ibis Re II Ltd. 2012-1 B; Mythen Re Ltd. 2012-1

Name	Date of Issue	Maturity	Spread (at issuance)	Expected loss	Conditional expected loss	Prob. of 1st loss	Prob. of last loss
Successor X Ltd. 2012-1	01/2012	01/2015	1100	2.59%	83%	3.12%	2.24%
Ibis Re II Ltd. 2012-1 A	01/2012	02/2015	835	1.38%	59,2%	2.33%	0.89%
Ibis Re II Ltd. 2012-1 B	01/2012	02/2015	1350	3.38%	67,9%	4.98%	2.36%
Mythen Ltd. 2012-1	05/2012	05/2015	850	1.09%	73,6%	1.48%	0.82%

Notes: See Lane (2013) for Mythen Ltd. 2012-1 and Lane (2012) for Ibis Re II Ltd. and Successor X Ltd.

As in the last section, prices are given as the (annual) spread over LIBOR and the quarterly secondary market spreads are displayed in Figure 3. U.S. hurricanes can only occur during the hurricane season from June to November, such that these cat bonds cover the same seasons and we can assume that they have the same maturity and risk exposure period. Due to sub- and super-hedging considerations, prices at every point in time are ordered according to the related expected loss, i.e., a cat bond with higher expected loss generally has a higher spread.

Figure 3: Secondary market spreads for Successor X Ltd 2012-1, Ibis Re II Ltd. 2012-1 A, Ibis Re II Ltd. 2012-1 B and Mythen Re Ltd. 2012-1 from 06/30/12 to 03/31/14



Source of data: Lane (2013) and Lane (2014).

Analogously to the ILW pricing formulas in Equations (7) and (10), the price of cat bonds (in terms of the spread premium) is given by

$$V_{cat,T'}^{b,Y,T}(0) = V_{cat,T'}^{b,\tilde{Y},T}(0) + \left(V_{cat,T'}^{b,Y,T}(0) - V_{cat,T'}^{b,\tilde{Y},T}(0) \right), \quad (16)$$

where we observe the price of a binary cat bond with trigger level \tilde{Y} ($V_{cat,T'}^{b,\tilde{Y},T}(0)$) in the market and, analogously to Section 4.3, use an empirically calibrated model to calculate the difference term

$$D_{\tilde{Y}}^Y = V_{cat,T'}^{b,Y,T}(0) - V_{cat,T'}^{b,\tilde{Y},T}(0).$$

We exemplarily follow Galeotti, Guertler and Winkelvos (2012) and apply the model of Wang (2004) to calculate $D_{\tilde{Y}}^Y$, i.e.

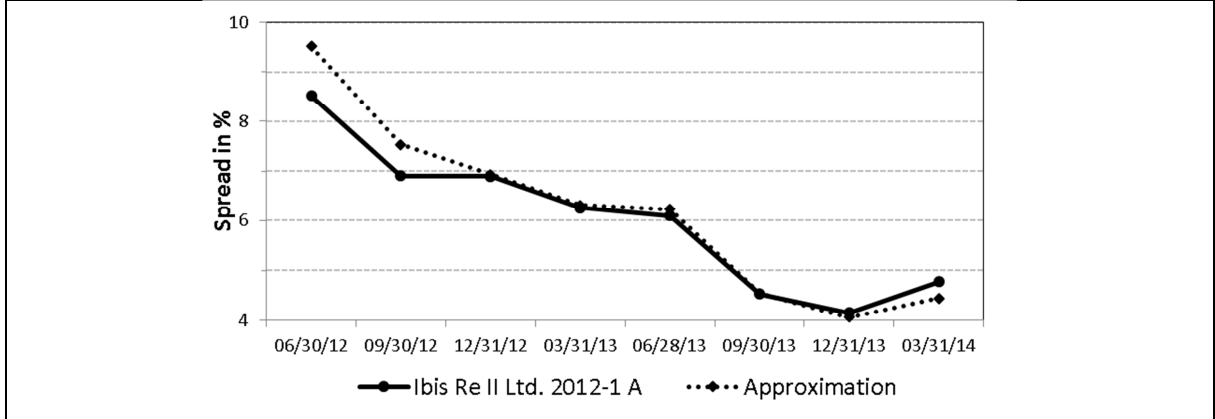
$$\begin{aligned} D_{\tilde{Y}}^Y &= V_{cat,T'}^{b,Y,T}(0) - V_{cat,T'}^{b,\tilde{Y},T}(0) \\ &= \frac{1}{2} \left(\Phi \left(\Phi^{-1} (PFL^Y) + \lambda \right) + \Phi \left(\Phi^{-1} (PLL^Y) + \lambda \right) - \left(\Phi \left(\Phi^{-1} (PFL^{\tilde{Y}}) + \lambda \right) + \Phi \left(\Phi^{-1} (PLL^{\tilde{Y}}) + \lambda \right) \right) \right), \end{aligned}$$

where PFL is the probability of the first loss, PLL the probability of the last loss and the market price of risk λ is obtained by solving the nonlinear regression

$$V_{cat,T'}^{b,\tilde{Y},T}(0) = \frac{1}{2} \left(\Phi \left(\Phi^{-1} (PFL^{\tilde{Y}_i}) + \lambda \right) + \Phi \left(\Phi^{-1} (PLL^{\tilde{Y}_i}) + \lambda \right) \right) + \varepsilon_i.$$

With λ and the parameters stated in Table 2, $D_{\tilde{Y}}^Y$ and $V_{cat,T'}^{b,Y,T}(0)$ can be calculated using Equation (16) and be compared with the empirical prices. As an example, the real and approximated prices of Ibis Re II Ltd. are shown in Figure 4 (where λ is determined by using the empirical input parameters of the other three cat bonds in Table 2), where we can again observe a high degree of consistency. As mentioned before, by combining empirically observed prices and only using theoretical models for the difference term (as displayed by Equation (16)), model risk can be considerably reduced.

Figure 4: Real and approximated prices (using the Wang transform for the difference term) of Ibis Re II Ltd. 2012-1 A from 06/30/12 to 03/31/14 using Mythen Ltd. 2012-1



6. OUTLOOK: PRICING INDEMNITY-BASED (DOUBLE-TRIGGER) CONTRACTS

For pricing indemnity-based contracts by means of replication, the tradability of the company loss L^T must be examined. Toward this end, we provide first thoughts on how the company loss could be treated and suggest three approaches, which can be explored in future research. The payoff of an indemnity-based ILW with attachment point A , maximum payoff M , trigger level Y , risk exposure period until T and maturing at time T' is given by

$$P_{ILW,T'}^{A,M,Y,T} = \min\left(M, (L_{T'}^T - A)_+\right) 1_{\{I_{T'}^T > Y\}} = \left((L_{T'}^T - A)_+ - (L_{T'}^T - (A+M))_+ \right) 1_{\{I_{T'}^T > Y\}}, \quad (17)$$

where $(\cdot)_+ = \max(\cdot, 0)$. From Equations (2) and (17), one can see the option-like structure of ILWs. A binary ILW is a binary call on the industry loss index I^T , and an indemnity-based ILW corresponds to a call spread option on the company loss L^T with an additional trigger for the industry loss.

Independence between company loss and industry loss index

First, in case I^T and L^T are independent, the price of an indemnity-based ILW (Equation (17)) can be described by

$$\begin{aligned} V_{ILW,T'}^{A,M,Y,T}(0) &= E^Q \left(e^{-rT'} \left((L_{T'}^T - A)_+ - (L_{T'}^T - (A+M))_+ \right) 1_{\{I_{T'}^T > Y\}} \right) \\ &= \left(C_{T'}^{L^T, A}(0) - C_{T'}^{L^T, A+M}(0) \right) E^Q \left(e^{-rT'} 1_{\{I_{T'}^T > Y\}} \right), \end{aligned}$$

where $C_{T'}^{L^T, A}(0)$ is the price of a call option on the company loss L^T with strike price A maturing at time T' . For the treatment of the index, we refer to the previous sections, and

$C_T^{L^T, A}(0)$ can be calculated using actuarial pricing principles, for instance. In addition, we refer to Møller (2003) for the latter part, where indifference prices are calculated using, e.g., the traditional variance principle and assuming that the company loss is log-normally distributed. However, Møller (2003) considers contracts combining insurance and financial risk, where independence is a more realistic assumption. Still, even though independence between the index and the company loss would imply a substantial degree of basis risk and would thus be problematic regarding the risk management purpose of an indemnity-based index-linked catastrophic loss instrument, empirical results by Cummins, Lalonde and Phillips (2004) for the U.S. market show that especially small insurers exhibit very low correlations between their own losses and industry loss indices. The authors also show that hedging using a statewide industry loss index is only effective for the largest insurers in their sample. In addition, this calculation could provide a lower bound for an efficient price.

Approximating the company loss based on the stock price loss

Second, one could approximate the company loss L^T with the loss of the stock price, which in turn is assumed to be traded, thus allowing arbitrage-free valuation, i.e. through a function f ,

$$L_t^T \approx f\left((S_s)_{t \geq s \geq 0}\right),$$

where f satisfies mild regularity conditions (e.g., such that expected values can be derived), where the entire path of the stock price $0 \leq s \leq t$ is considered. The function f would provide the company loss L_t^T by just observing the stock price and, therefore, since the stock is tradable on a liquid market, one can revert to arbitrage-free pricing methods for the company loss. For instance, one can assume that the only events causing a jump of the stock price are catastrophic events. In this case, the function f should sum up all jumps until time t or the abnormal returns following a catastrophe for a specific event window. The event study literature may provide insight regarding the relationship between insured losses due to catastrophes and their impact of stock prices. For instance, Hagendorff, Hagendorff and Keasey (2014) and Lamb (1995) provide evidence for the U.S. market that insured losses caused by catastrophes are reflected in the stock price loss. Based on these results, an approximation of the insured loss based on stock price reactions should probably also include company characteristics (e.g. the insurer's loss exposure), the market environment (degree of competition and market premium level), and may also depend on the type of catastrophe.

Doing this would result in a structured risk management product, which can be treated as in Cox, Fairchild and Pedersen (2004) and by using arbitrage-free valuation. However, there are several problems related to the use of such structured risk management products. A joint-

stock company is needed and there is a risk of moral hazard, as companies may try to publish negative information or overestimated company loss data (which could be adjusted after the maturity of the instrument) if the issued product is triggered to make it more valuable. Furthermore, one fundamental problem is that the stock price of the company will decline in a less pronounced way than in cases without hedging. Thus, the stock price loss would not reflect the actual loss resulting from the catastrophe. Hanke and Pötzelberger (2003) formally illustrate this issue for the case of arbitrage-free prices of options on a company's own stock. They show that the resulting prices differ from prices obtained through classical option pricing theory and that ignoring this effect implies arbitrage opportunities.

Exploiting the dependence: functional relationship between I and L

Third, one could assume a functional relationship between the industry loss index and the company loss motivated by the typically high degree of dependence between these processes.²⁴ This approach reduces indemnity-based index-linked catastrophic loss instruments to non-indemnity-based instruments and allows applying the replication techniques developed previously. A high degree of dependence is realistic, since basis risk arises if the company loss and the industry loss are not fully dependent (see, e.g., Harrington and Niehaus (1999), Gatzert and Kellner (2011)). For instance, Cummins, Lalonde and Phillips (2004) show that 36% of Florida hurricane insurers could effectively use instruments based on a statewide index without being exposed to a high degree of basis risk, while smaller insurers are generally exposed to high basis risk (see also Harrington and Niehaus (1999)). Following these arguments, one can assume a functional relationship between L^T and I^T , i.e.,

$$L_t^T = g(I_t^T),$$

implying that the payoff (see Equation (1)) can be described as

$$P_{T'}^T = h(I_{T'}^T, L_{T'}^T) = h(I_{T'}^T, g(I_{T'}^T)),$$

which leads to an analogous situation as in the case of non-indemnity-based instruments, since the payoff only depends on the industry loss index at maturity. One simple example could be $g(x) = a \cdot x$, with $0 \leq a \leq 1$ representing an indication of the market share. Empirical studies on the relation between the insurer's losses and the industry loss index as reflected in the function g may be based on regression or correlation analysis using empirical data of catastrophe losses and individual insurer losses, for instance, as is done by Harrington and

²⁴ Alternatively, one can assume stochastic dependence to reflect the case where the company is not affected by a catastrophe.

Niehaus (1999) or Cummins, Lalonde and Phillips (2004). The function g and the accuracy of the approximation may thereby depend on the insurer's exposure with respect to respective catastrophe, the aggregated industry loss, and the line of business, amongst others.

7. SUMMARY

This paper presents new approaches and techniques of how prices for index-linked catastrophic loss instrument can be derived using arbitrage-free pricing by proposing replication techniques and approximations that aim to overcome the requirement of direct tradability of the underlying loss indices. In particular, contrary to traditional option pricing theory, one cannot necessarily assume that the underlying industry loss index is tradable itself; however, there may be a liquid market for certain derivatives, including cat bonds. This is of great relevance today in the academic literature, but it will be even more relevant in the future for the insurance industry and financial investors, when index-linked catastrophic loss instruments become even more widespread than today and when there are truly liquid markets for derivatives (e.g., cat bonds) on the industry loss index.

We apply the proposed approaches and considerations to the arbitrage-free pricing of ILWs, where we assume a liquid cat bond market to ensure the tradability of the underlying industry loss index. We hence consider a liquid index-linked cat bond market (equivalent to a liquid option market) for various trigger levels and maturities, and first derive prices for binary ILWs. We thereby show that we do not need any assumption concerning the distribution of the underlying industry loss index, which represents a major advantage as compared to other pricing approaches. If a suitable cat bond is not available for approximating ILW prices, we provide approximations under some reasonable assumptions. Thus, one main contribution of the paper is to propose approaches to overcome the crucial point of tradability of the underlying loss processes in case of index-linked catastrophic loss instruments through suitable approximations and by deriving explicit replicating portfolios or close approximations using traded derivatives providing explicit pricing formulas.

When calculating prices for indemnity-based double-trigger catastrophic loss instruments, which in addition to the index depend on the company loss, one major problem is the behavior of the company loss, which should be tradable in some sense when using risk-neutral valuation. We provide first considerations regarding three potential approaches for solving this issue and address problems associated with them, leaving further theoretical and empirical analyses for future research. In general, such considerations regarding indemnity triggers will be increasingly relevant in the future, especially against the background of an increasing share of indemnity-based cat bond transactions.

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Appendix

$I_t^{\tilde{T}}$ is a compound Poisson process defined through

$$I_t^{\tilde{T}} = \sum_{i=1}^{N_t} X^i,$$

where N_t and X^i are independent, N_t denotes the claims arrival process (i.e., a Poisson process with intensity λ) and X^i the i.i.d. claims. The probability for exceeding the trigger level is given through

$$Q(I_{\tilde{T}}^{\tilde{T}} > Y) = 1 - Q(I_{\tilde{T}}^{\tilde{T}} \leq Y) = 1 - e^{-\lambda \tilde{T}} \sum_{k=0}^{\infty} \frac{(\lambda \tilde{T})^k}{k!} Q\left(\sum_{i=1}^k X^i \leq Y\right). \quad (19)$$

Furthermore, we assume that the industry loss index incorporates the actual catastrophe losses instantaneously and if $k = 2$ or more catastrophes occur, the industry loss index almost surely exceeds the trigger level. Hence, we can approximate Equation (19) through (sum until $k = 1$)

$$Q(I_{\tilde{T}}^{\tilde{T}} > Y) \approx 1 - e^{-\lambda \tilde{T}} - e^{-\lambda \tilde{T}} \lambda \tilde{T} Q(X^1 \leq Y). \quad (20)$$

An interpretation is that there are just two possibilities for the index not to be triggered: no claim or one claim with size X^1 lower than the trigger level Y . This should approximately be true if we model only the occurrence of very large events. Solving this equation for $Q(X^1 \leq Y)$ leads to

$$Q(X^1 \leq Y) = \frac{-Q(I_{\tilde{T}}^{\tilde{T}} > Y) + 1 - e^{-\lambda \tilde{T}}}{e^{-\lambda \tilde{T}} \lambda \tilde{T}} = \frac{e^{r \tilde{T}} V_{cat, \tilde{T}'}^{b, 0, Y, \tilde{T}}(0) - e^{-\lambda \tilde{T}}}{e^{-\lambda \tilde{T}} \lambda \tilde{T}}.$$

We use $T = T'$, $\tilde{T} = \tilde{T}'$ and Equation (20) again to calculate $D_{\tilde{T}, \tilde{T}'}^{T, T'}$

$$\begin{aligned}
D_{\tilde{T},\tilde{T}}^{T,T} &= e^{-rT} E^Q \left(1_{\{I_T^T > Y\}} - 1_{\{I_{\tilde{T}}^{\tilde{T}} > Y\}} \right) = e^{-rT} \left(Q(I_T^T > Y) - Q(I_{\tilde{T}}^{\tilde{T}} > Y) \right) \\
&= e^{-rT} \left(1 - e^{-\lambda T} - e^{-\lambda T} \lambda T Q(X^1 \leq Y) - \left(1 - e^{-\lambda \tilde{T}} - e^{-\lambda \tilde{T}} \lambda \tilde{T} Q(X^1 \leq Y) \right) \right) \\
&= e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda T} \lambda T Q(X^1 \leq Y) + e^{-\lambda \tilde{T}} + e^{-\lambda \tilde{T}} \lambda \tilde{T} Q(X^1 \leq Y) \right) \\
&= e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda T} \lambda T \frac{e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}}}{e^{-\lambda \tilde{T}} \lambda \tilde{T}} + e^{-\lambda \tilde{T}} + e^{-\lambda \tilde{T}} \lambda \tilde{T} \frac{e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}}}{e^{-\lambda \tilde{T}} \lambda \tilde{T}} \right) \\
&= e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda(T-\tilde{T})} \frac{T}{\tilde{T}} \left(e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}} \right) + e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) \right).
\end{aligned}$$

Now we can calculate $V_{ILW,T}^{b,Y,T}(0)$ through:

$$\begin{aligned}
V_{ILW,T}^{b,Y,T}(0) &= e^{-rT} - e^{-r(T-\tilde{T})} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) + D_{\tilde{T},\tilde{T}}^{T,T} \\
&= e^{-rT} - e^{-r(T-\tilde{T})} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) + e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda(T-\tilde{T})} \frac{T}{\tilde{T}} \left(e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}} \right) + e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) \right) \\
&= e^{-rT} - e^{r(\tilde{T}-T)} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) + e^{r(\tilde{T}-T)} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) + e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda(T-\tilde{T})} \frac{T}{\tilde{T}} \left(e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}} \right) \right) \\
&= e^{-rT} + e^{-rT} \left(-e^{-\lambda T} - e^{-\lambda(T-\tilde{T})} \frac{T}{\tilde{T}} \left(e^{r\tilde{T}} V_{cat,\tilde{T}}^{b,0,Y,\tilde{T}}(0) - e^{-\lambda \tilde{T}} \right) \right)
\end{aligned}$$