Capacity Aversion

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Abstract

This paper examines an agent's asymmetric attitudes toward non-probabilistic subjective uncertainty. Our main theorems are generalizations of Arrow-Pratt approximation and the Jensen inequality under model uncertainty as described by the Choquet expected utility model of Schmeidler (1982,1989), Gilboa (1987) and Sarin and Wakker (1992). We show that the order of Arrow-Pratt approximation differs between gains and losses, which is not the case in the the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005). We propose the Jensen inequality for all concave functions and provide a new economic meaning to Schmeidler's uncertainty aversion by disentangling risk aversion from uncertainty aversion. These results are then applied to analyze the demand for insurance and public investment decisions.

Key words: Choquet integrals; Arrow-Pratt approximation; Jensen inequality; Demand for insurance; Public investment decisions

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1 Introduction

The concept of risk aversion measures the extent to which an agent dislikes risk in the expected utility (EU) framework. The basic tool for studying risk aversion is the well-known Jensen inequality. An agent is risk averse if and only if replacing a risky asset with its expected value makes her better off. The Jensen inequality states that this is true if and only if the agent's utility function is concave. One way to measure an agent's degree of risk aversion for a given risk is to ask her how much she is willing to pay to eliminate it. This payment is called the risk premium. For small risks, the risk premium is characterized by Arrow-Pratt approximation, which states that the risk premium is approximately proportional to the variance of the risk. As the size of the risk approaches zero, the risk premium approaches zero as the square of the risk size. Under the EU framework, this means that all agents are neutral with respect to small risks. This property is called second-order risk aversion (Segal and Spivak, 1990). One feature of risk aversion and Arrow-Pratt approximation is that, under EU, they are symmetric for both gains and losses. Mathematically, this means that risk aversion and the magnitude of Arrow-Pratt approximation are independent of the monotonicity of the utility function.

Although the EU model is a cornerstone of modern research, it has received some criticism. One criticism is that the EU model does not account for model uncertainty. Therefore, some studies exame the agent's attitudes toward risk and uncertainty. Maccheroni, Marinacci and Ruffion (2013) (hereafter MMR) propose the analogue of the classic ArrowPratt approximation under model uncertainty as described by the smooth model of ambiguity of Klibanoff, Marinacci, and Mukerji (2005) (hereafter KMM). Lang (2017) defines first-order and second-order ambiguity aversion which are independent of specific uncertainty models.

In this paper, we investigate whether and how Arrow-Pratt approximation and the Jensen inequality might need to be modified when agents have imprecise, non-probabilistic beliefs about uncertainty. Specifically, we consider EU for a nonadditive subjective probability measure (Schmeidler 1982,1989; Gilboa 1987; Sarin and Wakker 1992), which captures Knightian uncertainty (Knight, 1921). We use Choquet expected utility (CEU), whereby agents have beliefs represented by capacities (nonadditive subjective probabilities). The standard Choquet integral is characterized as an asymmetric integral, which means that the integral behaves asymmetrically when multiplied by a negative quantity compared to when it is multiplied by a positive quantity. Therefore, we may anticipate that CEU-type decision makers will exhibit asymmetric behaviors.

One way to measure an agent's degree of aversion to a capacity is to ask her how much she is willing to pay to eliminate it. In this study, the answer to this question is referred to as the capacity premium. We propose definitions of this capacity premium and show that when an agent faces a gain, she exhibits second-order capacity aversion, and when an agent faces a loss, she exhibits first-order capacity aversion. Our result demonstrates that, under CEU, the capacity premium approximation is asymmetric in magnitude for gains and losses. MMR (2013, Proposition 3) show that both risk premium and ambiguity premium are second-order in the frame work of smooth model of decision making under KMM (2005). This is quite different from the CEU model. Therefore, the CEU model can provide us additional insights.

The present version of Choquet integrals of the Jensen inequality requires that the agent's utility function is increasing and concave (see Marinacci and Montrucchio 2004, Proposition 4.14; Asano 2006, Theorem 4), and the assumption of increasing utility is an essential aspect of their proofs. However, an agent has decreasing utility for a loss when she faces a risk of loss. Therefore, the current version of the Jensen inequality cannot apply to such a setting. In this paper, we propose a Jensen inequality when the agent's utility function is concave. In this setting, we find that concavity (risk aversion) alone is insufficient to yield the Jensen inequality. In fact, we need to consider uncertainty aversion as proposed by Schmeidler (1989). This result provides new economic meaning for Schmeidler's uncertainty aversion.

We investigate differences in the outcomes of the demand for insurance and public investment decisions when we assume that agents have imprecise non-probabilistic beliefs about uncertainty rather than being EU decision makers. In Schmeidler's framework, we first show that a risk-averse agent will purchase full insurance at an actuarially fair price if and only if she is uncertainty averse. Then, we show that the Arrow-Lind Principle, which holds in an economy in which all agents are of the von-Neumann and Morgenstern type, is maintained in an economy in which agents behave according to CEU, provided that all agents are winners. However, this is not the case when all agents are losers. We then consider an economy that consists of winners and losers. Our result shows that the Arrow-Lind Principle still holds if the number of losers is small relative to the number of winners.

This work proceeds as follows. Section 2 discusses some of the results regarding CEU. Sections 3 and 4 investigate capacity premium approximation and the Jensen inequality, respectively. Section 5 discusses the applications. Section 6 concludes the paper.

2 Choquet expected utility

Because this paper is devoted to an agent's behavior toward subjective uncertainty based on the CEU paradigm, we first review this approach.

A capacity μ on Ω is a real-valued set function that satisfies the following conditions:

(i) $\mu(\Phi) = 0$.

(ii) $\mu(A) \leq \mu(B)$ if $A \subseteq B \subseteq \Omega$.

(iii) $\mu(\Omega) = 1$.

Let $X : \Omega \to R$ be a function. Define $G_{\mu,X}(x) = \mu(X > x)$, and the pseudo-inverse function $\check{G}_{\mu,X}$ of $G_{\mu,X}(x)$ the quantile function of X with respect to μ. The Choquet integral of X with respect to μ is defined as

Definition 2.1. *(Denneberg 1994, p61) The Choquet integral of X, which is defined in Ω, is*

$$\int X d\mu = \int_0^1 \check{G}_{\mu,X}(t)dt, \tag{1}$$

where the right-hand side (RHS) is a Riemann integral.

Schmeidler (1989), Gilboa (1987), and Sarin and Wakker (1992) propose an axiomatized choice theory: in this CEU model, an agent with utility function u uses the Choquet integral as a choice criterion. In this paper, we assume that all derivatives of u are continuous.

Definition 2.2. *The CEU of u with respect to μ is*

$$\int u(X) d\mu = \int_0^1 \check{G}_{\mu,u(X)}(t)dt. \tag{2}$$

We follow Denneberg (1994).

Lemma 2.3. *(Denneberg 1994, Proposition 4.1) When $u(x)$ is an increasing function,*

$$\check{G}_{\mu,u(X)} = u \circ \check{G}_{\mu,X}. \tag{3}$$

Thus, when u is an increasing function, the CEU of u with respect to capacity μ is

$$\int u(X) d\mu = \int_0^1 u \circ \check{G}_{\mu,X}(t)dt. \tag{4}$$

Applying the same approach to Lemma 2.3, it is clear that when $u(x)$ is decreasing in x, $u\circ(-\check{G}_{\mu,X}) = \check{G}_{\mu,u(-X)}$, i.e., $u\circ(-\check{G}_{\mu,-X}) = \check{G}_{\mu,u(X)}$.[1] We obtain the following result:

Lemma 2.4. *When $u(x)$ is a decreasing function, $\check{G}_{\mu,u(X)} = u\circ(-\check{G}_{\mu,-X})$.*

Therefore, when u is a decreasing function, the CEU of u with respect to capacity μ is

$$\int u(X)d\mu = \int_0^1 u\circ(-\check{G}_{\mu,-X}(t))dt. \tag{5}$$

(4) and (5) indicate that CEU is asymmetric for increasing and decreasing utility functions. Hence, CEU-type agents may behave asymmetrically toward gains and losses. The merit of (4) and (5) is that they provide us with tools to investigate the decisions of CEU-type agents in a tractable Riemann integral paradigm.

3 First- and second-order capacity aversion

One way to measure an agent's degree of capacity aversion for kX is to ask her how much she is willing to pay to eliminate kX, where k is a positive number to measure the scale of the capacity. The answer to this question is referred to as the capacity premium $\pi(k)$. We propose the following definitions of capacity premium.

Definition 3.1. *When u is increasing and assuming w_0 is the fixed initial wealth, the capacity premium $\pi_I(k)$ ($\pi_D(k)$) for kX is defined by*

$$\int u(w_0 + kX)d\mu = u(w_0 + k\int Xd\mu - \pi_I(k)). \tag{6}$$

$$\int u(w_0 - kX)d\mu = u(w_0 - k\int Xd\mu - \pi_D(k)), \tag{7}$$

$\pi_I(k)$ ($\pi_D(k)$) may be positive, which is the case when the agent is capacity averse, or negative, which is the case when the agent is capacity loving. In the following proposition, we derive some conclusions concerning agents' behavior based on the notations in the above definitions.

[1]Specifically, for a decreasing function $u(x)$, if u and $G_{\mu,X}$ are one-to-one functions, we obtain

$$G_{\mu,u(-X)}(y) = \mu(u(-X) > y) = \mu(X > -u^{-1}(y)) = G_{\mu,X}(-u^{-1}(y)) = G_{\mu,X}\circ(-u^{-1}(y)).$$

Thus, $\check{G}_{\mu,X}(G_{\mu,u(-X)}(y)) = -u^{-1}(y)$ and $u\circ(-\check{G}_{\mu,X}(G_{\mu,u(-X)}(y))) = y$. Let $G_{\mu,u(-X)}(y) = z$; then, $y = \check{G}_{\mu,u(-X)}(z)$. We obtain $u\circ(-\check{G}_{\mu,X}(z)) = \check{G}_{\mu,u(-X)}(z)$.

Proposition 3.2.

$$\pi_I(k) = -\frac{u''(w_0)}{u'(w_0)}[\int_0^1 [\breve{G}_{\mu,X}(t)]^2 dt - [\int_0^1 \breve{G}_{\mu,X}(t)dt]^2]k^2 + o(k^2)$$

and

$$\pi_D(k) = (-\int X d\mu - \int -X d\mu)k + o(k).$$

Proof. See Appendix. □

$\pi_I(k)$ involves two terms. The first is the Arrow-Pratt absolute risk-aversion coefficient $-\frac{u''}{u'}$. The second term, $\int_0^1 [\breve{G}_{\mu,X}(t)]^2 dt - [\int_0^1 \breve{G}_{\mu,X}(t)dt]^2$, measures the agent's attitude toward uncertainty. It can be shown that $\int_0^1 [\breve{G}_{\mu,X}(t)]^2 dt - [\int_0^1 \breve{G}_{\mu,X}(t)dt]^2 \geq 0$. $F(t)$ is defined as the uniform probability distribution function on $[0,1]$. Then

$$\begin{aligned} & \int_0^1 [\breve{G}_{\mu,X}(t)]^2 dt - [\int_0^1 \breve{G}_{\mu,X}(t)dt]^2 \\ = \ & \mathbf{E}_F[\breve{G}_{\mu,X}(t)^2] - \mathbf{E}_F[\breve{G}_{\mu,X}(\tilde{t})]\mathbf{E}_F[\breve{G}_{\mu,X}(t)] \\ = \ & \mathbf{Cov}_F[\breve{G}_{\mu,X}(t), \breve{G}_{\mu,X}(t)] \\ \geq \ & 0 \quad (\textit{Since } \breve{G}_{\mu,X}(t) \textit{ is decreasing in } t), \end{aligned} \tag{8}$$

where $\mathbf{E}_F(\cdot)$ and $\mathbf{Cov}_F(\cdot)$ are the conditional expectation and conditional covariance operators, which are Riemann integrals.

As the size k of this uncertainty approaches zero, the capacity premium approaches k^2. Hence we obtain an attitude capacity aversion of order 2, which means, for a small capacity, the agent is almost neutral.

Note that Schmeidler's uncertainty aversion implies that $\int X d\mu + \int -X d\mu \leq 0$; hence, $\pi_D(k)$ is negative when the agent is uncertainty averse as indicated by Schmeidler. As the size k of this uncertainty approaches zero, the premium approaches k. This property is called "first-order capacity aversion"[2].

[2] If the capacity premium $\pi_D(k)$ for kX is defined

$$\int u(w_0 - kX)d\mu = u(w_0 + k\int -X d\mu - \pi_D(k)), \tag{9}$$

then, as a direct corollary of Proposition 3.2, we obtain

$$\pi_D(k) = -\frac{u''(w_0)}{u'(w_0)}[\int_0^1 [\breve{G}_{\mu,-X}(t)]^2 dt - [\int_0^1 \breve{G}_{\mu,-X}(t)dt]^2]k^2 + o(k^2). \tag{10}$$

That is, we are unable to obtain "first-order capacity aversion".

Proposition 3.2 shows that the orders of capacity aversion for gains and losses are asymmetric in magnitude. For a gain, the agent is neutral for small capacities. However, for a loss, small capacities cannot be ignored.

MMR (2013, Proposition 3) extend the ArrowPratt approximation to account for model uncertainty as described by KMM (2005). Their result shows a second order effect in evaluation for both gains and losses. Proposition 3.2 indicates that CEU, a canonical model, would expect results we are unable to obtain from the smooth ambiguity model.

4 The Jensen inequality for Choquet integrals

We first recall a version of Choquet integrals of the classic Jensen inequality.

Proposition 4.1. *(Marinacci and Montrucchio 2004, Proposition 4.14; Asano 2006,Theorem 4) Suppose that* u *is increasing and concave; then,* $\int u(X)d\mu \leq u(\int X d\mu)$ *for all* μ*.*

We know the $\int u(X)d\mu \leq u(\int X d\mu)$ for all concave u within the standard Jensen inequality under EU, but it is unable to be obtained by Proposition 4.1. Therefore, it is useful to develop a Jensen inequality for all risk-averse agents. The following proposition presents the result.

Proposition 4.2. *The following two statements are equivalent:*

(i) $\int u(X)d\mu \leq u(\int X d\mu)$ *for all* u *such that* $u'' \leq 0$*:*

(ii) $\int X d\mu + \int -X d\mu \leq 0$.

Proof. See Appendix. □

Following an idea proposed by Schmeidler (1989) and Dow and Werlang (1992), we define $C_{\mu,X}(x) = 1 - G_{\mu,-X}(-x) - G_{\mu,X}(x)$. $C_{\mu,X}(x)$ is a natural measure of imprecise beliefs, which is widely used in the literature on nonadditive measures (see, e.g., Walley 1991). When $C_{\mu,X}(x) \geq 0$, we describe the agent as uncertainty averse. Dow and Werlang (1992) show that $\int X d\mu + \int -X d\mu = -\int C_{\mu,X}(t)dt$. Hence, Schmeidler's uncertainty aversion implies $\int X d\mu + \int -X d\mu \leq 0$. They also show that, for capacities u and v, $C_{\mu,X} \geq C_{\nu,X}$ if and only if $\int X d\mu + \int -X d\mu \leq \int X d\nu + \int -X d\nu$.

Proposition 4.1 states that when an agent faces a risk of a gain, risk aversion alone is sufficient to lead to the Jensen inequality. However, Proposition 4.2 states that when she faces a risk of a loss, risk aversion alone is insufficient to lead to the Jensen inequality. In fact, we need both risk aversion and uncertainty aversion as indicated by Schmeidler.

Proposition 4.1 shows that risk aversion implies capacity aversion for an agent with an increasing utility function. Proposition 4.2 states that capacity aversion consists of two different aversions: risk aversion and uncertainty aversion. First, one considers the risk using the concavity of u. Second, one evaluates the uncertainty via the convexity of the capacity[3]. We thus attach economic meaning to Schmeidler's uncertainty aversion. This result improves our formal understanding of uncertainty aversion by disentangling risk aversion from uncertainty aversion.

5 Applications

5.1 The demand for insurance

Mossin (1968) first demonstrates that an EU agent will prefer to purchase full insurance at an actuarially fair price. Here, we re-examine Mossin's result in the CEU framework. We consider an insured with utility function u who faces potential monetary losses $L > 0$. L is a capacity. We define $\alpha \in [0, 1]$ as the share of losses to be insured. The insured is offered a fair insurance contract against L, and the insurance premium is $\alpha \int L d\mu$. The insured's payoff is

$$p(\alpha) = w - (1 - \alpha)L - \alpha \int L d\mu, \tag{11}$$

where w is her fixed initial wealth. If the insured purchases full insurance ($\alpha = 1$), then her payoff is $p(1) = w - \int L d\mu$. We also note that $\int L d\mu = \int [(1 - \alpha)L + \alpha \int L d\mu] d\mu$. Therefore, $\alpha = 1$ is the optimal action if and only if for all $\alpha \in [0, 1]$,

$$\begin{aligned} \int u(w - (1 - \alpha)L - \alpha \int L d\mu) d\mu &\leq u(p(1)) \\ &= u(w - \int L d\mu) \\ &= u(w - \int [(1 - \alpha)L + \alpha \int L d\mu] d\mu). \end{aligned} \tag{12}$$

From Proposition 4.2, we obtain the following result.

Proposition 5.1. *For $\alpha \in [0, 1]$, the following statements are equivalent.*

(i) $\alpha = 1$ is the optimal action for all risk-averse insureds;

(ii) $\int L d\mu + \int -L d\mu \leq 0$.

[3] Schmeidler shows that a convex capacity implies that $C_{\mu,X}(x) \geq 0$

Proposition 5.1 states that for a risk averse insured, full insurance is optimal if and only if $\int Ld\mu + \int -Ld\mu \leq 0$. The result shows that under CEU, full insurance might not be optimal even if the insured is risk averse. Full insurance is optimal when risk aversion and uncertainty aversion are both effective.

As the findings indicate that insureds often prefer complete insurance, even though all insurance companies have a loading premium, it is important to explain such behaviors (see, e.g., Doherty and Eeckhoudt, 1995 or Dupuis and Langlais 1997). Suppose that $\lambda > 0$ is the loading factor. Then the insured's payoff is

$$p(\alpha) = w - (1-\alpha)L - \alpha(1+\lambda)\int Ld\mu. \tag{13}$$

The insured's objective is to choose α to maximize

$$\int u(w - (1-\alpha)L - \alpha(1+\lambda)\int Ld\mu)d\mu. \tag{14}$$

We propose the following result:

Proposition 5.2. *Suppose that $u' \geq 0$ and $u'' \leq 0$. If the price of insurance includes a positive premium loading ($\lambda > 0$), then $\alpha = 1$ is the optimal action if and only if*

$$(1+\lambda)\int Ld\mu + \int -Ld\mu \leq 0. \tag{15}$$

Proof. See Appendix. □

Proposition 5.2 states that an insured can purchase full insurance even if the premium is not fair, as long as her subjective belief satisfies inequality (15). Note that (15) can be re-written as

$$\lambda \leq \frac{-[\int Ld\mu + \int -Ld\mu]}{\int Ld\mu}. \tag{16}$$

$\int Ld\mu + \int -Ld\mu$ measures the uncertainty aversion. The higher the uncertainty aversion is, the higher the $-[\int Ld\mu + \int -Ld\mu]$ is. We define $\frac{-[\int Ld\mu + \int -Ld\mu]}{\int Ld\mu}$ as *uncertainty aversion to anticipated loss ratio*. Then we can explain Proposition 5.2 in the following way: a risk averse insured will buy full insurance if and only if her *uncertainty aversion to anticipated loss* ratio is higher than the premium loading.

5.2 Public investment decisions

Arrow and Lind (1970) investigate the implications of risk for public investment decisions. They consider the case in which all individuals have the same preferences u, and their disposable incomes are identically distributed random variables represented by Y. Suppose that the government undertakes an investment with returns represented by X, which are independent of Y. Consider a specific taxpayer, and denote her percentage of this investment by s with $0 \leq s \leq 1$. Suppose that each taxpayer has the same tax rate and that there are n taxpayers, then $s = \frac{1}{n}$. Arrow and Lind (1970) show that

$$\mathbf{E}u(Y + \frac{X}{n}) = \mathbf{E}u(Y + \frac{1}{n}EX - \pi(n)), \tag{17}$$

where $\pi(n)$ is the risk premium of the representative individual, and $\mathbf{E}(\cdot)$ is the expectation operator, which is a Riemann integral. They demonstrate that when $n \to \infty$, $\pi(n)$ vanishes, as does the total of the risk premiums for all individuals, then $n\pi(n)$ approaches zero as n rises. This result implies that the total cost of risk-bearing ($n\pi(n)$) goes to zero as the population of taxpayers increases. Hence, if the population is large enough, its behavior toward risk should be almost risk neutral. Hence, a cost-benefit analysis can ignore investment risk.

Now, we reconsider the Arrow-Lind model within the CEU framework. We assume that there are n agents in the economy, which consists of n_I agents who gain $\frac{X}{n}$ and n_D agents who lose $\frac{X}{n}$, such that $n_I + n_N = n$. We also assume that the return on the public investment X is a random variable with nonadditive distribution μ. The prospect premiums $\pi_I(n)$ and $\pi_D(n)$ can be defined as follows:

$$\int \mathbf{E}u(Y + \frac{X}{n})d\mu = \mathbf{E}u(Y + \frac{\int X d\mu}{n} - \pi_I(n)) \tag{18}$$

and

$$\int \mathbf{E}u(Y - \frac{X}{n})d\mu = \mathbf{E}u(Y - \frac{\int X d\mu}{n} - \pi_D(n)). \tag{19}$$

Following Kihlstrom et al., (1981) and Nachman (1982), we define indirect utility function as

$$v(x) = \mathbf{E}u(x + Y).$$

Then, we have

$$\int v(\frac{X}{n})d\mu = v(\frac{\int X d\mu}{n} - \pi_I(n)) \tag{20}$$

and

$$\int v(-\frac{X}{n})d\mu = v(-\frac{\int X d\mu}{n} - \pi_D(n)). \tag{21}$$

We define a measure of the order of magnitude of the population proportions when the population is large enough.

Definition 5.3. *(i) n_I and n_D are equal in population if and only if $\frac{n_D}{n_I} = O(1)$;*

(ii) n_I is larger than n_D in population if and only if $\frac{n_D}{n_I} = O(\frac{1}{n^s})$, for $s > 0$.

We provide the following examples to illustrate the above definitions:

1. Example of population equivalence: $n_I = \frac{2}{3}n$ and $n_D = \frac{1}{3}n$.
2. Example of population dominance: $n_I = n - \sqrt{n}$, $n_D = \sqrt{n}$.

We define $\Pi(n) = n_I\pi_I(n) + n_D\pi_D(n)$ as the total cost of bearing X. From Proposition 3.2, we obtain the following result:

Proposition 5.4. *(i) If $n_I = n$, then $\Pi(n)$ goes to zero as $n \to \infty$;*

(ii) if $n_D = n$, then, as $n \to \infty$, $\Pi(n) \neq 0$;

(iii) if n_I and n_D are equal in population, then, as $n \to \infty$, $\Pi(n) \neq 0$; and

(iv) if n_I is larger than n_D in population, then $\Pi(n)$ goes to zero as $n \to \infty$.

Proof. See Appendix. □

Proposition 5.4 shows that the aggregate of premiums approaches zero in an economy in which all individuals are of the von-Neumann and Morgenstern type, which also holds in an economy in which agents behave according to CEU, provided that all the agents' utilities are increasing in the investment profits. This result is no longer valid when agents' utilities are decreasing in the investment profits. In an economy that consists of both types of agents, the Arrow-Lind principle is maintained if n_I is small relative to n_D as a share of the population. The intuition for our conclusions is that in a sufficiently large economy with two types of agents, public investment uncertainty can be ignored only when second-order capacity-averse agents will share in the uncertainty. However, if everyone satisfies first-order capacity aversion, then everyone should carry some uncertainties. When both types appear to exist, the total cost of risk-bearing goes to zero if the population of the second-order capacity-averse type is significantly larger than that of the other type.

6 Conclusion

This paper focuses on an agent's asymmetric attitudes toward non-additive subjective uncertainty. Our main theorems are generalizations of Arrow-Pratt approximation and the Jensen inequality in a CEU setting. In the models developed in this paper, the degree of Arrow-Pratt approximation is second order for a gain and first order for a loss; the Jensen inequality is maintained when the agent is both risk averse and uncertainty averse. Since Arrow-Pratt approximation and the Jensen inequality have potentially widespread implications for economics and finance analysis, further research on this topic is needed to develop applications beyond the demand for insurance and public investment decisions.

7 Appendix

7.1 Proof of Proposition 3.2

Suppose that u is increasing. From

$$\begin{aligned}\int u(w_0+kX)d\mu &= \int_0^1 \breve{G}_{\mu,u(w_0+kX)}(t)dt \quad (by\ (1)) \\ &= \int_0^1 u(w_0+k\breve{G}_{\mu,X}(t))dt, \quad (by\ (4)) \end{aligned} \tag{22}$$

we obtain

$$\int_0^1 u(w_0+k\breve{G}_{\mu,X}(t))dt = u(w_0+k\int Xd\mu-\pi_I(k)). \tag{23}$$

Fully differentiating (23) with respect to k, we obtain

$$\int_0^1 \breve{G}_{\mu,X}(t)u'(w_0+k\breve{G}_{\mu,X}(t))dt = (\int Xd\mu-\pi_I'(k))u'(w_0+k\int Xd\mu-\pi_I(k)). \tag{24}$$

Because $\pi_I(0)=0$, we obtain

$$u'(w_0)\int_0^1 \breve{G}_{\mu,X}(t)dt = (\int Xd\mu-\pi_I'(0))u'(w_0); \tag{25}$$

hence, $\pi_I'(0)=0$. Fully differentiating (24) with respect to k, we obtain

$$\begin{aligned}&\int_0^1 [\breve{G}_{\mu,X}(t)]^2u''(w_0+k\breve{G}_{\mu,X}(t))dt \\ =\ &(\int Xd\mu-\pi_I'(k))^2u''(w_0+k\int Xd\mu-\pi_I(k))-\pi_I''(k)u'(w_0+k\int Xd\mu-\pi_I(k)).\end{aligned} \tag{26}$$

Because $\pi_I(0)=\pi_I'(0)=0$, we obtain

$$u''(w_0)\int_0^1 [\breve{G}_{\mu,X}(t)]^2dt = (\int Xd\mu)^2u''(w_0)-\pi_I''(0)u'(w_0). \tag{27}$$

Hence, $\pi_I''(0) = -\frac{u''(w_0)}{u'(w_0)}[\int_0^1[\breve{G}_{\mu,X}(t)]^2dt-(\int Xd\mu)^2] = -\frac{u''(w_0)}{u'(w_0)}[\int_0^1[\breve{G}_{\mu,X}(t)]^2dt-[\int_0^1 \breve{G}_{\mu,X}(t)dt]^2]$.

From

$$\begin{aligned}\int u(w_0-kX)d\mu &= \int_0^1 \breve{G}_{\mu,u(w_0-kX)}(t)dt \quad (by\ (1)) \\ &= \int_0^1 u(w_0+k\breve{G}_{\mu,-X}(t))dt, \quad (by\ (5)) \end{aligned} \tag{28}$$

we obtain

$$\int_0^1 u(w_0 + k\breve{G}_{\mu,-X}(t))dt = u(w_0 - k\int X d\mu - \pi_D(k)). \quad (29)$$

Fully differentiating (29) with respect to k, we obtain

$$\int_0^1 \breve{G}_{\mu,-X}(t)u'(w_0 + k\breve{G}_{\mu,-X}(t))dt = (-\int X d\mu - \pi_D'(k))u'(w_0 - k\int X d\mu - \pi_D(k)). \quad (30)$$

Because $\pi_D(0) = 0$, we obtain

$$u'(w_0)\int_0^1 \breve{G}_{\mu,-X}(t)dt = (-\int X d\mu - \pi_D'(0))u'(w_0), \quad (31)$$

that is,

$$\int -X d\mu = -\int X d\mu - \pi_D'(0); \quad (32)$$

hence, we obtain $\pi_D'(0) = -\int X d\mu - \int -X d\mu$.

7.2 Proof of Proposition 4.2

First, let us recall three basic properties of a Choquet integral (see Denneberg 1994, Proposition 5.1):

(P1) $\int cX d\mu = c\int X d\mu$ for all $c \geq 0$;

(P2) $\int (X + c)d\mu = \int X d\mu + c$ for all $c \in R$; and

(P3) $X \leq Y$ implies $\int X d\mu \leq \int Y d\mu$.

(i) implies (ii):

Since u is concave, we obtain

$$\begin{aligned} u(X) - u(\int X d\mu) &\leq u'(\int X d\mu)(X - \int X d\mu) \quad (33) \\ &= -u'(\int X d\mu)(\int X d\mu - X). \quad (34) \end{aligned}$$

There are two cases: either $u'(\int X d\mu) \geq 0$ or $u'(\int X d\mu) < 0$.

Case 1: $u'(\int X d\mu) \geq 0$.

Taking the Choquet integral on both sides of (33), we obtain

$$\begin{aligned} \int u(X)d\mu - u(\int X d\mu) &\leq \int u'(\int X d\mu)(X - \int X d\mu)d\mu \quad \textit{(by (P2) and (P3))} \\ &= u'(\int X d\mu)\int (X - \int X d\mu)d\mu \quad \textit{(by } u' \geq 0 \textit{ and (P1))} \\ &= u'(\int X d\mu)(\int X d\mu - \int X d\mu) = 0. \quad \textit{(by (P2))} \quad (35) \end{aligned}$$

Case 2: $u'(\int X d\mu) < 0$.

Taking the Choquet integral on both sides, we obtain

$$\begin{aligned}\int u(X)d\mu - u(\int Xd\mu) &\leq \int -u'(\int Xd\mu)(\int Xd\mu - X)d\mu \quad (by\ (P2)\ and\ (P3)) \\ &= -u'(\int Xd\mu)\int(\int Xd\mu - X)d\mu \quad (by\ u'<0\ and\ (P1)) \\ &= -u'(\int Xd\mu)(\int Xd\mu + \int -Xd\mu). \quad (by\ (P2)) \end{aligned} \tag{36}$$

From $\int Xd\mu + \int -Xd\mu \leq 0$, we obtain $\int u(X)d\mu \leq u(\int Xd\mu)$.

(ii) implies (i)

Suppose that $\int Xd\mu + \int -Xd\mu > 0$ and $\int u(kX)d\mu \leq u(k\int Xd\mu)$. For v with $v' > 0$, define

$$\int v(-kX)d\mu = v(-k\int Xd\mu - \pi_D(k)). \tag{37}$$

From Proposition 3.2, we know that $\pi_D(k) = (-\int -Xd\mu - \int Xd\mu)k + o(k)$. Hence, when k is small enough, $\pi_D(k) < 0$, which implies that $\int v(-kX)d\mu > v(-k\int Xd\mu)$. Define $u(x) = v(-x)$, then we have $\int u(kX)d\mu > u(k\int Xd\mu)$, which indicates that the result is a contradiction!

7.3 Proof of Proposition 5.2

From

$$\begin{aligned} M(\alpha) &= \int u(w - (1-\alpha)L - \alpha(1+\lambda)\int Ld\mu)d\mu \\ &= \int_0^1 u(w + (1-\alpha)\breve{G}_{\mu,-L}(t) - \alpha(1+\lambda)\int Ld\mu)dt \quad (by\ (5)) \end{aligned} \tag{38}$$

we obtain the first-order condition

$$M'(\alpha) = \int_0^1 [-\breve{G}_{\mu,-L}(t) - (1+\lambda)\int Ld\mu]u'(w + (1-\alpha)\breve{G}_{\mu,-L}(t) - \alpha(1+\lambda)\int Ld\mu)dt = 0 \tag{39}$$

and the second-order condition

$$M''(\alpha) = \int_0^1 [-\breve{G}_{\mu,-L}(t) - (1+\lambda)\int Ld\mu]^2 u''(w + (1-\alpha)\breve{G}_{\mu,-L}(t) - \alpha(1+\lambda)\int Ld\mu)dt \leq 0, \tag{40}$$

Evaluating the $M'(\alpha)$ at $\alpha = 1$ shows that

$$\begin{aligned} M'(1) &= u'(w - (1+\lambda)\int Ld\mu)\int_0^1 [-\breve{G}_{\mu,-L}(t) - (1+\lambda)\int Ld\mu]dt \\ &= u'(w - (1+\lambda)\int Ld\mu)[-\int_0^1 \breve{G}_{\mu,-L}(t)dt - (1+\lambda)\int Ld\mu] \\ &= u'(w - (1+\lambda)\int Ld\mu)[-\int -Ld\mu - (1+\lambda)\int Ld\mu]. \end{aligned} \tag{41}$$

Since $u' \geq 0$, the sign of $M'(1)$ will be positive (negative) if and only if

$$\int -Ld\mu + (1+\lambda)\int Ld\mu \leq 0.$$

We conclude the proof.

7.4 Proof of Proposition 5.4

(i) and (ii): These are the results obtained by directly applying Proposition 3.2.

(iii) and (iv):

From Proposition 3.2, we obtain

$$\begin{aligned} \Pi(n) &= n_I \pi_I(n) + n_D \pi_D(n) \\ &= n_I O(\frac{1}{n^2}) + n_D O(\frac{1}{n}) \\ &= \frac{n_I}{n_I + n_D} O(\frac{1}{n}) + \frac{n_D}{n_I + n_D} O(1) \\ &= \frac{1}{1 + \frac{n_D}{n_I}} O(\frac{1}{n}) + \frac{\frac{n_D}{n_I}}{\frac{n_D}{n_I} + 1} O(1). \end{aligned} \tag{42}$$

When $\frac{n_D}{n_I} = O(1)$, then $\Pi(n) = O(1)$.

When $\frac{n_D}{n_I} = O(\frac{1}{n^s})$, for $s > 0$, $\Pi(n) = O(\frac{1}{n})$.

8 References

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