Consumption and Portfolio Choice under Internal Multiplicative Habit Formation

Servaas van Bilsen
Dept. of Quantitative Economics
University of Amsterdam
and NETSPAR

A. Lans Bovenberg
Department of Economics
Tilburg University
CentER and NETSPAR

Roger J. A. Laeven†
Dept. of Quantitative Economics
University of Amsterdam
EURANDOM and CentER

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Abstract

This paper explores the optimal consumption and investment behavior of an individual who derives utility from the ratio between his consumption and an endogenous habit. We obtain closed-form policies under general utility and stochastic investment opportunities, by developing a non-trivial linearization to the budget constraint. This enables us to explicitly characterize how habit formation affects the optimal shock absorbing mechanism and stock-bond investments. We also show that in a setting which combines habit formation with stochastic differential utility consumption no longer grows at unrealistically high rates at high ages and investments in risky assets decrease.


Keywords: Internal Habit Formation, Stochastic Differential Utility, Approximation Technique, Return Smoothing, Life Cycle Investment.

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†Corresponding author. Mailing Address: PO Box 15867, 1001 NJ Amsterdam, The Netherlands. Phone: +31 (0) 20 525 4219.
1 Introduction

The internal habit formation literature can be subdivided along two main model specifications that have been widely used in Finance and Macroeconomics: additive habits (Constantinides (1990)) and multiplicative habits (Abel (1990)). The latter model specification, also referred to as the ratio model of habit formation, is advocated in particular by Carroll (2000).

The endogenous nature of the habit in internal habit formation models substantially complicates the analysis of optimal consumption and portfolio policies and associated asset pricing problems. In important work, Schroder and Skiadas (2002) show how to transform models with internal additive habits into models without habit formation, enabling closed-form solutions to a wide range of asset pricing problems involving additive habits. Conversely, their approach allows to translate solutions to familiar consumption and portfolio choice problems under time-separable or recursive utility into solutions to corresponding problems exhibiting additive habit formation. So far, however, internal habit formation models with multiplicative habits cannot be solved analytically. Thus, analysis of the appealing ratio habit model necessarily resorts to numerical methods to obtain solutions, impeding their applicability.

In this paper, we develop a closed-form approximation approach to solve consumption and portfolio choice problems involving internal multiplicative habits and apply it to analyze how multiplicative habit formation affects the conventional optimal consumption and investment dynamics in three important cases. In a nutshell, our approach consists of first applying a change of variables, redefining consumption in relative terms, and next pursuing a suitable linearization of the static budget constraint. Our approximation approach transforms consumption and portfolio problems with ratio habits into approximate consumption and portfolio problems without habit. This enables us to obtain closed-form approximate solutions to a variety of consumption and portfolio problems with multiplicative habit formation under general utility functionals and stochastic investment opportunities.

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1 Another dimension along which the habit formation literature can be subdivided is internal habit (Sundaresan (1989), Abel (1990), and Constantinides (1990)) and external habit (the catching-up-with-the-Joneses specification of Abel (1990), Campbell and Cochrane (1999) and Chan and Kogan (2002)).

2 Additive habits are also referred to as linear habits or the difference model of habit formation.

3 See, e.g., Van Bilsen, Laeven, and Nijman (2017) who employ Schroder and Skiadas (2002) to explicitly derive the optimal consumption and portfolio policies under loss aversion and endogenous updating of the reference level—two key features of prospect theory (Tversky and Kahneman (1992)).

4 Linearization of the static budget constraint is not uncommon in the economics literature; see, in a different context, e.g., Campbell and Mankiw (1991), and Fuhrer (2000).
We show that our approximation approach is very accurate when consumption is not too far from the habit level, and when the habit level is not very persistent. Our numerical results show that the approximation error, when measured in terms of the relative decline in certainty equivalent consumption, is typically less than 1%, and that the explicit optimal policies to the approximate problems closely mimic the numerically evaluated optimal policies to the original problems. Having closed-form solutions has three key advantages: they reveal the roles played by the various model parameters, they are readily amenable to comparative statics analysis, and they facilitate the implementation of the optimal consumption and investment policies.

We apply our general approach to three cases of internal multiplicative habit formation: a base model with time-separable utility in terms of relative consumption, constant relative risk aversion, and constant investment opportunities; a first extension with stochastic investment opportunities involving stochastic interest rates; and a second extension with stochastic differential utility. We can summarize our three main findings as follows. First, we characterize in explicit closed-form how an individual under the base model adjusts both his current consumption level and future growth rates of consumption following a stock return shock. We show that the features of the optimal shock absorbing mechanism and investment strategy are characterized by two preference parameters: the coefficient of relative risk aversion and the strength of habit persistence. These preference parameters not only have clear economic interpretations by themselves but also induce clearly interpretable implications for the optimal consumption and portfolio decisions: the coefficient of relative risk aversion determines how large the impact of a stock return shock is on the individual’s current consumption level and the strength of habit persistence determines the impact of a stock return shock on future growth rates of consumption. We also find that current consumption is less volatile than the underlying investment portfolio. In other words, the individual can take on substantial stock market risk without negatively impacting his year-on-year consumption.

\[5\] In particular, we show that a stock return shock has a smaller impact on the current consumption level of a highly risk averse individual than on that of a weakly risk averse individual.

\[6\] We find that the more persistent the habit level is, the larger the impact of a stock return shock on future growth rates of consumption will be.

\[7\] We argue that the optimal policies provide a preference-based justification for the existence of (recently developed) annuity products in which surpluses earned in good years support benefit payouts in bad years. Such annuity products have been analyzed by, e.g., Guillén, Jørgensen, and Nielsen (2006), Jørgensen and Linnemann (2012), Guillén, Nielsen, Pérez-Marín, and Petersen (2013), Maurer, Rogalla, and Siegelin (2013a), Linnemann, Bruhn, and Steffensen (2014), and Maurer, Mitchell, Rogalla, and Siegelin (2016).
consumption volatility. Furthermore, we show that the individual implements an investment strategy that is nearly independent of the state of the economy (especially at high ages) and depends only on age.

Second, we find that in an extension of the base model that allows for stochastic interest rates and stock-bond investments the optimal hedging demand for bonds is hump-shaped over the life cycle. Two counteracting forces determine the life cycle pattern for the optimal hedging demand for bonds. On the one hand, the individual is less willing to substitute consumption over time as he grows older. This causes the hedging bond portfolio weight to increase with age. On the other hand, the impact of an interest rate shock on the price of future consumption is larger the younger the individual is. This causes the hedging bond portfolio weight to decrease with age.

A general feature of many habit formation models (including our base model) is that median consumption grows at unrealistically high rates (especially at high ages) except when the time discount rate is excessive. We therefore develop an extended preference model that combines stochastic differential utility (SDU) with the ratio model of habit formation. Thus, the elasticity of intertemporal substitution is decoupled from the coefficient of relative risk aversion. Our third main finding is, then, that in this setting, quite remarkably, habit formation does not necessarily lead to unrealistically high unconditional median growth rates of consumption at the end of life, even when the time discount rate is moderate. Furthermore, wealth accumulation is substantially lower under this extended model than under the base model. Hence, an individual whose preferences combine habit formation with SDU invests less wealth in the stock market compared to an individual without SDU. To the best of our knowledge, a model that combines SDU with the ratio model of habit formation has not yet been studied in existing literature. The closest to the current paper in this respect is Schroder and Skiadas (1999) who analytically studied SDU but did not consider the ratio model of

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8This finding stands in sharp contrast to the design of standard unit-linked insurance products and traditional drawdown strategies in which a more aggressive portfolio strategy directly translates into a higher year-on-year consumption volatility. See, e.g., Dus, Maurer, and Mitchell (2005), Horneff, Maurer, Mitchell, and Dus (2008), and Maurer, Mitchell, Rogalla, and Kartashov (2013b) for a description of these products.

9More specifically, the individual lowers the share of his investment portfolio invested in the risky stock as he becomes older. Indeed, the available time to adjust current and future consumption levels in response to a stock return shock declines with age.

10This may explain why not many young individuals include long-term bonds in their investment portfolios.

Duffie and Epstein (1992) introduced the notion of SDU as a continuous-time limit of the preference models studied by Kreps and Porteus (1978) and by Epstein and Zin (1989).
habit formation.

The problem of optimal consumption and portfolio choice over the life cycle has intrigued many authors since the seminal work of Merton (1969, 1971), Mossin (1968) and Samuelson (1969). Their work has been extended along many dimensions. Many life cycle consumption and portfolio choice papers assume a standard preference model; that is, decision makers’ preferences are described by either CRRA utility or Epstein-Zin utility. Although these standard preference models satisfy a set of desirable axioms, their ability to describe how people actually make decisions under risk is known to be limited. Furthermore, their predictions fail to explain well-documented facts about actual consumption and portfolio behavior. The shortcomings of standard preference models have inspired many researchers to develop alternative theories for decision-making under risk, including habit formation.

Several authors have explored the implications of these alternative preference theories for optimal investment decisions or intertemporal consumption behavior. Most relevant to our base model are Detemple and Zapatero (1991, 1992), Schroder and Skiadas (2002), Bodie, Detemple, Otruba, and Walter (2004) and Munk (2008) who analyze the optimal consumption and investment behavior of an individual who derives utility from the difference between consumption and the internal habit level, rather than some ratio of these as we do. Contrary to the ratio model of habit formation, the optimal consumption choice implied by the difference model of habit formation exceeds the habit level in each economic scenario. This so-called addictive behavior of consumption is criticized theoretically e.g., by Chapman (1998) and Carroll (2000), and arguably at odds with empirical evidence. Finally, we note that the ratio model of habit formation has been employed in other papers to analyze monetary policy (Fuhrer (2000)), asset prices with an external habit (Abel (1999), Chan and Kogan

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12 For instance, to accommodate time-varying investment opportunities (see, e.g., Campbell, Cocco, Gomes, Maenhout, and Viceira (2001), Wachter (2002), Chacko and Viceira (2005), Liu (2007), and Laeven and Stadje (2014)); uncertain labor income (see, e.g., Viceira (2001), Cocco, Gomes, and Maenhout (2005), and Gomes and Michaud (2005)); housing costs (see, e.g., Cocco (2005), and Yao and Zhang (2005)); and unexpected health expenditures (see, e.g., Edwards (2008)).

13 Among the most notable alternatives are prospect theory (Kahneman and Tversky (1979), and Tversky and Kahneman (1992)), regret theory (Loomes and Sugden (1982), Bell (1982, 1983), Sugden (1993), and Quiggin (1994)), disappointment (aversion) theory (Bell (1985), Loomes and Sugden (1986), and Gul (1991)), and habit formation (Abel (1990), Constantinides (1990) and Sundaresan (1989)).


15 For instance, Crossley, Low, and O’Dea (2013) show that consumption levels declined significantly during recent recessions, contradicting the addictive property of consumption.
and Gómez, Priestley, and Zapatero (2009) and an internal habit (Smith and Zhang (2007)), macroeconomic growth (Carroll, Overland, and Weil (1997), Carroll, Overland, and Weil (2000) and Carroll (2000)), and portfolio choice with uninsurable labor income risk (Gomes and Michaelides (2003)).

2 Model

2.1 Asset Prices, Pricing Kernel and Budget Constraint

Denote by $T > 0$ a fixed terminal time. We represent the randomness in the economy by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which is defined a standard $N$-dimensional Brownian motion $\{W_t\}_{0 \leq t \leq T}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $\{W_t\}_{0 \leq t \leq T}$. Throughout, (in)equalities between random variables hold $\mathbb{P}$-almost surely.

We consider a financial market consisting of an instantaneously risk-free asset and $N$ risky assets. Trading takes place continuously over $[0, T]$. The price of the risk-free asset, $B_t$, satisfies

$$\frac{dB_t}{B_t} = r_t \, dt, \quad B_0 = 1.$$  \hfill (1)

We assume that the scalar-valued risk-free rate process, $\{r_t\}_{0 \leq t \leq T}$, is $\mathcal{F}_t$-progressively measurable and satisfies $\int_0^T |r_t| \, dt < \infty$. The $N$-dimensional vector of risky asset prices, $S_t$, obeys the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t, \quad S_0 = 1_N.$$  \hfill (2)

Here, $1_N$ represents an $N$-dimensional vector consisting of all ones. We assume that the $N$-dimensional mean rate of return process, $\{\mu_t\}_{0 \leq t \leq T}$, and the $(N \times N)$-matrix-valued volatility process, $\{\sigma_t\}_{0 \leq t \leq T}$, are $\mathcal{F}_t$-progressively measurable and satisfy $\int_0^T ||\mu_t|| \, dt < \infty$ and $\sum_{i=1}^N \sum_{j=1}^N \int_0^T (\sigma_t)_{ij}^2 \, dt < \infty$, respectively.

We impose the following additional condition on $\sigma_t$. For some $\epsilon > 0$,

$$\zeta^T \sigma_t \sigma_t^T \zeta \geq \epsilon ||\zeta||^2, \quad \text{for all } \zeta \in \mathbb{R}^N,$$  \hfill (3)

where $\top$ denotes the transpose sign. Condition (3) implies in particular that $\sigma_t$ is invertible.
The $\mathcal{F}_t$-progressively measurable market price of risk process, $\{\lambda_t\}_{0 \leq t \leq T}$, satisfies
\[
\sigma_t \lambda_t = \mu_t - r_t 1_N.
\] (4)

The unique positive-valued state price density process, $\{M_t\}_{0 \leq t \leq T}$, is given by (see, e.g., Karatzas and Shreve (1998)):
\[
M_t = \exp \left\{ -\int_0^t r_s \, ds - \int_0^t \lambda_s^\top \, dW_s - \frac{1}{2} \int_0^t ||\lambda_s||^2 \, ds \right\}.
\] (5)

The economy consists of a single individual endowed with initial wealth $A_0 \geq 0$. This individual chooses an $\mathcal{F}_t$-progressively measurable $N$-dimensional portfolio process $\{\pi_t\}_{0 \leq t \leq T}$ (representing the dollar amounts invested in the $N$ risky assets) and an $\mathcal{F}_t$-progressively measurable consumption process $\{c_t\}_{0 \leq t \leq T}$ so as to maximize lifetime utility. We impose the following integrability conditions on the portfolio and consumption processes:
\[
\int_0^T \pi_t^\top \sigma_t d\pi_t < \infty, \quad \int_0^T |\pi_t (\mu_t - r_t 1_N)| \, dt < \infty, \quad \mathbb{E} \left[ \int_0^T |c_t|^r \, dt \right] < \infty \quad \forall \, r \in \mathbb{R}.
\] (6)

The wealth process, $\{A_t\}_{0 \leq t \leq T}$, satisfies the following dynamic budget constraint:
\[
dA_t = (r_t A_t + \pi_t^\top \sigma_t \lambda_t - c_t) \, dt + \pi_t^\top \sigma_t \, dW_t, \quad A_0 \geq 0 \text{ given.}
\] (7)

We call a consumption-portfolio strategy $\{c_t, \pi_t\}_{0 \leq t \leq T}$ admissible if the associated wealth process is positive.

2.2 Habit Level

Denote by $h_t$ the individual’s habit level at time $t$. Following Kozicki and Tinsley (2002) and Corrado and Holly (2011), we assume that the log habit level $\log h_t$ satisfies the following dynamic equation:
\[
d \log h_t = (\beta \log c_t - \alpha \log h_t) \, dt, \quad \log h_0 = 0.
\] (8)

We normalize the initial log habit $\log h_0$ to zero i.e., $h_0$ equals unity. The preference parameter $\alpha \geq 0$ represents the rate at which the log habit level exponentially depreciates. When $\alpha$ is small, the log habit level exhibits a high degree of memory. The preference
Parameter $\beta \geq 0$ models the relative importance between the initial habit level and the individual’s past consumption choices. When $\beta$ is large, the individual’s past consumption choices are relatively important, i.e., the habit level is relatively persistent. We impose the following restriction on the individual’s preference parameters:

$$\alpha \geq \beta. \tag{9}$$

The parameter restriction (9) prevents the individual’s habit level from growing exponentially over time; see Eqn. (15) below.

We can write the log habit level $\log h_t$ as a weighted sum of the individual’s log past consumption choices:

$$\log h_t = \beta \int_0^t \exp \left\{-\alpha (t - s) \right\} \log c_s \, ds. \tag{10}$$

The log habit level $\log h_t$ is additive (linear) in past levels of log consumption. Corrado and Holly (2011) show that for the ratio model of habit formation, the habit specification (10) is more desirable than an arithmetic habit specification in which the habit level $h_t$ itself is additive in past levels of consumption.

### 2.3 Dynamic Optimization Problem

Let $U(c/h) \in \mathbb{R} \cup \{-\infty\}$ be the individual’s utility derived from the process $c/h = \{c_t/h_t\}_{0 \leq t \leq T}$, representing the ratio between consumption and the habit level. We place no restrictions on $U$. The individual now faces the following dynamic optimization problem:

$$\max_{c_t, \pi_t: 0 \leq t \leq T} U\left(\frac{c_t}{h_t}\right)$$

subject to:

$$\begin{align*}
dA_t &= \left( r_t A_t + \pi_t^T \sigma_t \lambda_t - c_t \right) dt + \pi_t^T \sigma_t dW_t, \\
d \log h_t &= (\beta \log c_t - \alpha \log h_t) dt,
\end{align*} \tag{11}$$

with $A_0 \geq 0$ given.

Section 3 presents a solution technique for analytically solving (11) based on developing a linearization to the individual’s budget constraint.
3 Solution Method

3.1 An Equivalent Problem

We can, by virtue of the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989, 1991)), transform the individual’s dynamic optimization problem (11) into the following equivalent static variational problem:

\[
\max_{c_t, 0 \leq t \leq T} U \left( \frac{c_t}{h_t} \right) \quad \text{s.t.} \quad \mathbb{E} \left[ \int_0^T M_t c_t \, dt \right] \leq A_0,
\]

where \( M_t \) is given by (5). After the optimal consumption choice \( c_t^{\text{opt}} \) has been determined, one can determine the optimal portfolio choice \( \pi_t^{\text{opt}} \) using hedging arguments.

3.2 A Change of Variable Transformation

Denote by \( \hat{c}_t \) the ratio between the individual’s consumption choice and his habit level; that is,

\[
\hat{c}_t = \frac{c_t}{h_t}.
\]

(13)

We can express the dynamics of the log habit level in terms of the individual’s log relative consumption choice \( \log \hat{c}_t = \log (c_t/h_t) \) as follows (the dynamics of the log habit level follow from substituting \( \log c_t = \log h_t + \log \hat{c}_t \) into (8)):

\[
d \log h_t = (\beta \log \hat{c}_t - [\alpha - \beta] \log h_t) \, dt.
\]

(14)

Hence, the individual’s log habit level \( \log h_t \) is explicitly given by

\[
\log h_t = \beta \int_0^t \exp \left\{ -(\alpha - \beta)(t - s) \right\} \log \hat{c}_s \, ds.
\]

(15)

Eqn. (15) shows that because of the parameter restriction \( \alpha \geq \beta \) (see (9)), the individual’s habit level is prevented from growing exponentially over time.

We can thus rewrite the individual’s static optimization problem (12) in terms of
\( \hat{c}_t = c_t / h_t \) yielding the following equivalent problem:

\[
\begin{align*}
\max_{\hat{c}_t: 0 \leq t \leq T} & \quad U(\hat{c}) \\
\text{s.t.} & \quad \mathbb{E} \left[ \int_0^T M_t h_t \hat{c}_t \, dt \right] \leq A_0,
\end{align*}
\]

(16)

We then obtain the optimal consumption choice \( c_t^{\text{opt}} \) from the optimal relative consumption choice \( \hat{c}_t^{\text{opt}} \) as follows:

\[
c_t^{\text{opt}} = h_t^{\text{opt}} \hat{c}_t^{\text{opt}}.
\]

(17)

To solve the individual’s static optimization problem (12), we can thus restrict ourselves to solving (16). In applications, it is still typically impossible to solve the individual’s static optimization problem (16) analytically. The reason for this is that the new static budget constraint in (16) depends non-linearly on the individual’s relative consumption choices. Section 3.3 develops a linearization for the new static budget constraint in (16). After applying this linearization, we are able to obtain an analytical closed-form expression for the individual’s consumption policy in a wide range of interesting cases.

### 3.3 Linearization of the New Static Budget Constraint

This section presents a linear approximation to the left-hand side of the new static budget constraint in (16) around the constant relative consumption trajectory \( \{\hat{c}_t\}_{0 \leq t \leq T} = 1 \).

We expect that the approximation error is relatively small when the individual’s consumption choice is not too far from the habit level, and when \( \beta \) is small or \( \alpha \) is large. We expect that the approximation error is relatively large when the individual’s consumption choice deviates much from the habit level. Section 6 explores the approximation error induced by applying our linearization to the new static budget constraint in (16). The numerical results reveal that the approximation error is typically less than 1% in terms of relative decline in certainty equivalent consumption.

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\(^{16}\) We can determine \( h_t^{\text{opt}} \) by substituting the optimal past relative consumption choices \( \hat{c}_s^{\text{opt}} (s \leq t) \) into (15).

\(^{17}\) Indeed, substitution of the habit level \( h_t \) (see (15)) into the budget constraint in (16) shows that the new static budget constraint in (16) is non-linear in the individual’s relative consumption choices.

\(^{18}\) The Appendix considers the more general case in which the budget constraint is approximated around the constant relative consumption trajectory \( \{\hat{c}_t\}_{0 \leq t \leq T} = x \) for some positive \( x \).
and that our closed-form approximated strategies closely mimic the genuinely optimal (but numerically evaluated) strategies.

Appendix A proves the following theorem.

**Theorem 3.1.** Consider an individual who aims to solve the optimization problem (16). This problem is equivalent to the following simpler problem up to a first-order approximation of the static budget constraint:

$$\max_{\hat{\epsilon}, 0 \leq t \leq T} U(\hat{\epsilon})$$

subject to

$$\mathbb{E} \left[ \int_0^T \hat{M}_t \hat{\epsilon}_t \, dt \right] \leq \hat{A}_0.$$  \hspace{1cm} (18)

Here,

$$\hat{M}_t = M_t (1 + \beta P_t)$$  \hspace{1cm} (19)

with $P_t$ defined as:

$$P_t = \mathbb{E}_t \left[ \int_t^T \frac{M_s}{M_t} e^{-(\alpha-\beta)(s-t)} \, ds \right].$$  \hspace{1cm} (20)

The quantity $\hat{A}_0$ denotes the individual’s initial wealth associated with the approximate problem (18). We determine the individual’s initial wealth $\hat{A}_0$ such that the approximate optimal consumption strategy $\{\hat{\epsilon}_t\}_{0 \leq t \leq T} = \{h_t^* \hat{\epsilon}_t^*\}_{0 \leq t \leq T}$ is budget-feasible.\(^{19}\)

The relative consumption choice $\hat{\epsilon}_t^*$ solving (18) is an approximation to the optimal relative consumption choice $\hat{\epsilon}_t^{opt}$. Note that the endogenous habit level $h_t$ now does not appear in (18), thanks to the change of variable transformation and, crucially, our linearization of the static budget constraint.

**Remark 1.** Using a suitable transformation, Schroder and Skiadas (2002) convert models with a linear internal habit and utility expressed as the difference between consumption and habit into models without habit formation. Interestingly and quite surprisingly (to us), the transformed state-price density process (19) and hence the transformed problem (18) are identical to the transformed counterparts in Schroder and Skiadas (2003), the difference being that $\hat{\epsilon}_t$ represents relative consumption $\epsilon_t/h_t$ in our framework while it represents surplus consumption $\epsilon_t - h_t$ in their framework. In their setting, the original

\(^{19}\)Here, $h_t^*$ denotes the habit level at time $t$ implied by substituting the approximate optimal past relative consumption choices $\hat{\epsilon}_s^* (s \leq t)$ into (15).
budget constraint is equivalent to the budget constraint in the transformed problem:
\[
E \left[ \int_0^T M_t c_t \, dt \right] = E \left[ \int_0^T \hat{M}_t (c_t - h_t) \, dt \right] + K_1,
\]
while in our setting the new budget constraint first-order approximates the original budget constraint:
\[
E \left[ \int_0^T M_t c_t \, dt \right] \approx E \left[ \int_0^T \hat{M}_t \frac{c_t}{h_t} \, dt \right] + K_2.
\]
Here, \( K_1 \) and \( K_2 \) are constants that are irrelevant in determining the first-order optimality conditions. Note also that the interpretation of the parameters \( \alpha \) and \( \beta \) is different in the two papers as we consider the dynamics of the log habit level \( \log h_t \) where they consider the dynamics of the habit level \( h_t \).

4 Constant Relative Risk Aversion Utility

In this section, we assume that lifetime utility is defined as follows:
\[
U \left( \frac{c}{h} \right) = E \left[ \int_0^T e^{-\delta t} \frac{1}{1 - \gamma} \left( \frac{c_t}{h_t} \right)^{1-\gamma} \, dt \right],
\]
with \( h_t \) given by (10). Here, \( E \) represents the unconditional expectation, \( \delta \geq 0 \) stands for the individual’s subjective rate of time preference, and \( \gamma > 0 \) denotes the individual’s coefficient of relative risk aversion. Specification (23) corresponds to the habit formation model proposed by Abel (1990).

In (23), relative risk aversion is constant. Several authors explore the implications of the difference model of habit formation in which relative risk aversion is not constant but rather depends on surplus consumption \( c_t - h_t \). As a result, the optimal strategies under the difference model of habit formation are fundamentally different from the optimal strategies under the ratio model of habit formation. In particular, the portfolio strategy of an individual whose preferences are represented by the difference model of habit formation heavily depends on the individual’s endogenous habit level. In our model, by contrast, the portfolio strategy is nearly state-independent; see Section 4.5 for more details.

4.1 Optimal Consumption Choice

Theorem 4.1 presents the (approximate) optimal consumption choice \( c_t^* \).
Theorem 4.1. Consider an individual with lifetime utility (23) and habit formation process (8) who solves the consumption and portfolio choice problem (18). Denote by $h_t^*$ the habit level at time $t$ implied by substituting the (approximate) optimal past relative consumption choices $\hat{c}_s^*$ ($s \leq t$) into (15), and by $y$ the Lagrange multiplier associated with the static budget constraint in (18). Then the (approximate) optimal consumption choice $c_t^*$ is given by
\[ c_t^* = h_t^* \left( ye^{\delta t M_t} \right)^{-\frac{\gamma}{\delta}}. \] (24)

The Lagrange multiplier $y \geq 0$ is determined such that the individual’s original budget constraint holds with equality.

4.2 Sensitivity and Volatility of Future Consumption

In what follows (except Section 4.7), we assume that the economy consists of two assets: a risk-free asset and a risky stock. The dynamics of these two assets are assumed to be given by
\[ \frac{d B_t}{B_t} = r \, dt, \] (25)
\[ \frac{d S_t}{S_t} = \mu \, dt + \sigma \, dW_t. \] (26)

Denote by $q_{t-s}$ the sensitivity of current log consumption, $\log c_t^*$, to a past stock return shock, $\sigma \, dW_s$ ($s \leq t$). We find the explicit expression (see Appendix A)\footnote{We note that if $\alpha = \beta$, then (28) reduces to $Q_{t-s} = 1 + \beta(t-s)$.}
\[ q_{t-s} = \frac{\lambda}{\gamma \sigma} Q_{t-s}, \] (27)
with
\[ Q_{t-s} = 1 + \frac{\beta}{\alpha - \beta} \left[ 1 - \exp \left\{ -(\alpha - \beta)(t-s) \right\} \right]. \] (28)

The sensitivity $q_{t-s}$, dictating the optimal shock absorbing mechanism, depends on the time distance between the date at which the stock return shock occurs (i.e., time $s$) and the date of consumption (i.e., time $t > s$). In particular, a closer inspection of (27) reveals that $q_h$ increases with the time distance (or horizon) $h$: a current stock return shock has a smaller impact on log consumption in the near future (i.e., small $h$) than on log consumption in the distant future (i.e., large $h$). Our utility framework thus provides
a preference-based justification for the existence of annuity products in which current stock return shocks are not fully reflected into current annuity payouts. These products work as follows (see, e.g., Guillén et al. (2006), Linnemann et al. (2014), and Maurer et al. (2016)). In the case of a positive investment return, the annuity payout will go up by less than the realized return. The remaining investment gains will be added to a reserve fund. In the case of a negative investment return, the annuity payout will be protected and will decrease by a lower percentage than the realized return. This ‘payout protection’ will be paid from the reserve fund. The just described mechanism results in an excessively smooth and excessively sensitive payout stream.

The individual’s preference parameters $\gamma$, $\alpha$ and $\beta$ have clearly interpretable implications for the individual’s optimal consumption choice. We find that a current stock return shock $\sigma \, dW_t$ has a smaller impact on the current consumption level of a highly risk averse individual (i.e., high $\gamma$) than on that of a weakly risk averse individual (i.e., low $\gamma$). Indeed, a highly risky averse individual is more risk averse to year-on-year fluctuations in current consumption than a weakly risk averse individual. The coefficient $0 \leq \beta/\alpha \leq 1$, which measures the degree of habit persistence, determines the impact of a current stock return shock on the future growth rates of (median) consumption. If the individual’s preferences exhibit a large degree of habit persistence (i.e., $\beta/\alpha$ is close to one), a current stock return shock will have a relatively large impact on future growth rates of consumption: the individual adjusts the future growth rates of consumption downwards (upwards) by a relatively large percentage after the occurrence of a negative (positive) stock return shock. Figure 1(a) illustrates the sensitivity $q_h$ as a function of the horizon $h$ for various parameter values.

The annualized volatility of the future consumption choice of an individual with conventional CRRA utility (henceforth referred to as a CRRA individual) does not depend on the horizon $h$: consumption in the near future exhibits the same annualized volatility as consumption in the far future. By contrast, the future consumption choice of an individual whose preferences exhibit internal habit formation do depend on the horizon $h$. More specifically, we find that the annualized volatility of $\log c^*_t+h$ is given in closed-form by

$$\Sigma_h = \sqrt{V_t \left[ \log c^*_t+h \right]} = \sqrt{\int_0^h q_v^2 \, dv} \cdot \sigma. \quad (29)$$

Here, $V_t$ denotes the variance conditional on the information available at time $t$. Note that the annualized variance is proportional to the normalized integrated squared sensitivity $q_v$. Figure 1(b) shows the annualized volatility of future consumption as a function of
the horizon \( h \). As shown by this figure, for an individual with habit preferences, the annualized volatility of consumption in the near future is smaller than the annualized volatility of consumption in the far future.

[Place Figure 1 about here]

4.3 Shock Absorbing Mechanism

This section illustrates in more detail how the current and the future consumption levels of an individual whose preferences exhibit internal habit formation respond to an unexpected stock return shock. We consider an individual who starts working at the age of 25 and passes away at the age of 85. He invests and spends his accumulated wealth according to the ratio model of habit formation (23) with preference parameters \( \gamma = 10, \alpha = 0.3 \) and \( \beta = 0.3 \). For illustration purposes, the individual adjusts consumption only once a year.\(^{21}\) We also explore how a stock return shock affects the current and the future consumption levels of an individual whose preferences are characterized by conventional CRRA utility. The CRRA individual invests 50% of his accumulated wealth in the stock market (i.e., his relative risk aversion coefficient is equal to 2). This roughly coincides with the share of wealth a 58-year-old individual with habit preferences invests in the stock market; for more details on the portfolio strategy of an individual with habit preferences, see Section 4.5.

Figures 2(a) and (b) illustrate the impact of a 40% stock price decline in year one on current and future consumption choices. A CRRA individual fully translates a current stock return shock into his current consumption level. In this example, the current consumption level of a CRRA individual decreases by 17.80% after the stock price shock has been realized. The stock return shock does not affect the future growth rates of his consumption; see Figure 2(a) which shows that the shape of the median consumption path of a CRRA individual remains unaffected by a stock return shock. An individual whose preferences exhibit internal habit formation does not fully translate a current stock return shock into his current consumption level. As a result, the relative decline in the current consumption level of an individual with habit preferences is typically smaller than the relative decline in the current consumption level of a CRRA individual. In this example, his current consumption level drops by only 3.84% while the current

\(^{21}\)All figures and tables in this paper assume that the individual adjusts consumption only once a year. We note that this is not a restriction of our framework. We could also illustrate the case in which the individual adjusts consumption every month or every week.
consumption level of a CRRA individual drops by more than 17%. The flip side of protecting current consumption is that the shape of the median consumption path cannot remain unchanged following a stock return shock; see Figure 2(b) which shows that the individual adjusts the future growth rates of his median consumption downwards. A consequence of adjusting future growth rates of median consumption is that the gap between the median consumption choice before the occurrence of the stock return shock and the median consumption choice after the occurrence of the stock return shock widens with the horizon.

Figures 2(c) and (d) illustrate the impact of a 20% stock price increase in year two on current and future consumption choices. As in Figure 2(a), the CRRA individual directly absorbs the current stock return shock into his current consumption level. The current stock return shock has a smaller impact on the current consumption level of an individual with habit preferences than on that of the CRRA individual. Indeed, an individual with habit preferences has a strong preference to protect current consumption. In fact, in this example, he only consumes slightly more than last year, because he has translated part of last year’s (negative) stock return shock into consumption of this year. Furthermore, as a result of the current stock price increase, he adjusts the future growth rates of his median consumption upwards; see Figure 2(d).

4.4 Decomposition of the Consumption Dynamics

We can decompose the dynamics of the individual’s log consumption choice $\log c^*_t$ as follows (see Appendix A):

$$
\begin{align*}
  \mathrm{d} \log c^*_t &= \frac{1}{\gamma} \left( \hat{r}_t + \frac{1}{2} \lambda^2 - \delta \right) \mathrm{d}t + g_t \mathrm{d}t + \frac{\lambda}{\gamma \sigma} \sigma \mathrm{d}W_t. \\
  \hat{r}_t &= \beta + \frac{r - \alpha \beta P_t}{1 + \beta P_t}, \\
  g_t &= \frac{\mathrm{d}Q_t}{\mathrm{d}t} \log c^*_0 + \frac{1}{\gamma} \int_0^t \frac{\mathrm{d}Q_{t-s}}{\mathrm{d}t} \left( \hat{r}_s + \frac{1}{2} \lambda^2 - \delta \right) \mathrm{d}s + \frac{\lambda}{\gamma \sigma} \int_0^t \frac{\mathrm{d}Q_{t-s}}{\mathrm{d}t} \sigma \mathrm{d}W_s,
\end{align*}
$$

with $P_t$ defined in (20).

The right-hand side of Eqn. (30) consists of three terms. The first term represents the...
unconditional median growth rate of log consumption. This term models the utility gain and cost of postponing consumption. Lowering current consumption will increase utility because of two reasons: first, it dampens future habit levels, and second, it increases total expected investment gains (because the individual saves more). The first effect is captured by the preference parameters $\alpha$ and $\beta$, while the second effect is captured by the parameters $r$ and $\lambda$. A reduction in current consumption also implies a utility cost. Indeed, the individual is impatient: he prefers to consume sooner rather than later. This effect is captured by the preference parameter $\delta$. The term $g_t$ represents past stock return shocks that the individual translates into the current median growth rate of log consumption. This term disappears if the individual’s preferences do not exhibit internal habit formation (i.e., $\beta = 0$, so that $Q_h = 1$ for all $h$). Finally, the last term corresponds to the current stock return shock that the individual directly translates into his current consumption level.

Figure 3 illustrates a consumption path of an individual whose preferences exhibit internal habit formation. This figure assumes the same parameter values as in Section 4.3. As shown by Figure 3, the consumption stream of an individual with habit preferences is smoother than the consumption stream of a CRRA individual. As is well-known, an excessively smooth consumption stream is also consistent with aggregate consumption data (see, e.g., Flavin (1985), Deaton (1987), and Campbell and Deaton (1989)) and other behavioral models (see, e.g., K˝ oszegi and Rabin (2006, 2007, 2009), Pagel (2017), and Van Bilsen et al. (2017)).

[Place Figure 3 about here]

4.5 Optimal Portfolio Choice

Theorem 4.2 presents the (approximate) optimal portfolio choice $\pi_t^*$. Theorem 4.2. Consider an individual with lifetime utility (23) and habit formation process (8) who solves the consumption and portfolio choice problem (18). Then the (approximate) optimal portfolio choice $\pi_t^*$ is given by

$$\pi_t^* = \int_0^{T-t} q_h \frac{V_{t,h}}{V_t} dh \cdot A_t. \quad (33)$$

Here, $V_t = \int_0^{T-t} V_{t,h} dh$ and $V_{t,h}$ denotes the market value at time $t$ of $c_{t+h}^*$. Appendix A provides an explicit analytical expression for $V_{t,h}$ (see (67)).
Figure 4 illustrates the portfolio strategy $\pi_t^*/A_t$ of an individual with habit preferences. The individual implements a life cycle investment strategy: the share of accumulated wealth invested in the risky stock decreases as the individual ages. Indeed, the individual has less time to absorb a stock return shock as he grows older. A declining equity glide path during both the accumulation and the retirement phase is also commonly adopted by target date fund managers; see Morningstar (2017). The portfolio strategy of an individual with habit preferences stands in sharp contrast to the portfolio strategy of a CRRA individual. Such an individual implements an age-independent portfolio strategy; see the dotted line in Figure 4.

Figure 4 also shows that the portfolio strategy of an individual with habit preferences hardly varies with the state of the economy, especially at higher ages. The portfolio strategy is not completely state-independent: while the sensitivity $q_h$ and volatility $\Sigma_h$ of future consumption are fully state-independent due to the constant relative risk aversion property, a shock to the economy alters the shape of the median consumption stream (see Figure 2). In particular, long horizons benefit relatively more from a positive shock, while, on the other hand, short horizons suffer relatively less from a negative shock. As a result, the value weights $V_{t,h}/V_t$ in (33) change following a shock. However, this effect is small (second-order), so that the portfolio strategy is nearly insensitive to economic shocks.

Table 1 shows the (median) year-on-year volatility of accumulated wealth for various ages. The year-on-year consumption volatility is always equal to 2%, irrespective of the individual’s current age. With internal habit formation, the year-on-year consumption volatility is smaller than the year-on-year volatility of accumulated wealth. Hence, the individual can take substantial stock market risk without affecting the year-on-year consumption volatility. Indeed, the degree of habit persistence largely determines the share of accumulated wealth invested in the risky stock, while the individual’s coefficient of relative risk aversion largely determines the degree of variability of current

---

22 A state-independent portfolio strategy has three advantages for annuity providers. First, an annuity provider can implement the portfolio strategy without much effort: he does not have to monitor any state variables. Second, an annuity with a state-independent portfolio strategy is easy to communicate as the equity glide path is known at inception. Third, the individual typically achieves a prosperous expected payout stream at an affordable price. Indeed, if an annuity provider offers an annuity with a state-dependent portfolio strategy, then this portfolio strategy is often designed such that it protects customers against losses or locks in investment gains. While attractive from the viewpoint of avoiding losses, the flip side of this investment behavior is that upward potential can be rather limited.
consumption. As a result, given a certain degree of relative risk aversion, an individual with habit preferences invests more in the stock market early in life than an individual with conventional CRRA preferences. An individual with habit preferences translates a stock return shock not only in current consumption but also in future growth rates of consumption. This enables the individual to take a relatively risky position in the stock market at young ages.

4.6 Uncertain Date of Death

So far we have assumed that the terminal time $T$ is known at the beginning of the life cycle. However, the individual may also want to know how to drawdown his accumulated wealth if the terminal time $T$ is equal to his uncertain date of death. This section explores how an uncertain terminal time affects the consumption dynamics (30). We assume that the individual aims to maximize lifetime utility (23) where $T \geq 0$ now denotes the uncertain adult age at which the individual passes away.

We find that in this setting of uncertain terminal time, the individual’s log consumption choice $\log c_t^*$ evolves according to (see Appendix A)

$$
\frac{d \log c_t^*}{dt} = \frac{1}{\gamma} \left( \hat{r}_t + \frac{1}{2} \lambda^2 - \delta - H_t \right) dt + g_t dt + \frac{\lambda}{\gamma \sigma} dW_t,
$$

which is to be compared to (30). Here, $H_t$ denotes the force of mortality (hazard rate) at adult age $t$ (which is typically an increasing function of $t$). As shown by Eqn. (34), an increase in future consumption now implies two types of costs. First, the individual prefers to consume sooner rather than later. This effect is captured by the time preference rate $\delta$. Second, the individual may pass away before being able to enjoy future consumption. This effect is captured by the force of mortality $H_t$. As a result, the median consumption path is less steep compared to the case where the terminal time $T$ is assumed fixed; see Figure 5. In this figure, we compute the force of mortality using the unisex mortality table for the U.S. population for 2015.

23 We assume no uncertainty in the force of mortality.
4.7 Stochastic Interest Rate

This section explores the implications of a stochastic interest rate for the optimal portfolio choice of an individual with habit preferences. We assume that the economy consists of three assets: one (locally) risk-free asset, a risky stock and a zero-coupon bond with time to maturity $T_1$. The price of the risk-free asset, $B_t$, and the vector of risky asset prices, $S_t$, satisfy

$$\frac{dB_t}{B_t} = r_t \, dt,$$

$$(35)$$

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t,$$

$$(36)$$

where the risk-free interest rate $r_t$ follows an Ornstein-Uhlenbeck process, i.e.,

$$dr_t = \kappa (\bar{r} - r_t) \, dt + \sigma_r \rho \sqrt{1 - \rho^2} \, dW_t,$$

$$(37)$$

and $\mu_t$ and $\sigma_t$ are defined as follows:

$$\mu_t = \begin{bmatrix} r_t + \lambda_1 \sigma_S \\ r_t - \sigma_r D_{T_1} \left( \lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2} \right) \end{bmatrix},$$

$$(38)$$

$$\sigma_t = \begin{bmatrix} \sigma_S & 0 \\ -\sigma_r D_{T_1} \rho & -\sigma_r D_{T_1} \sqrt{1 - \rho^2} \end{bmatrix}.$$n

$$(39)$$

Here, $\kappa \geq 0$ denotes the mean reversion coefficient, $\bar{r}$ corresponds to the long-term interest rate, $\sigma_r > 0$ stands for the interest rate volatility, $-1 \leq \rho \leq 1$ models the correlation between the interest rate and the risky stock price, $\sigma_S > 0$ represents the stock return volatility, and $D_{T_1} = \frac{1}{\kappa} (1 - e^{-\kappa T_1})$ denotes the interest rate sensitivity of the bond. The market prices of risk associated with the two Brownian increments are given by $\lambda_1$ and $\lambda_2$. Appendix A proves the following theorem.

**Theorem 4.3.** Consider an individual with lifetime utility (23) and habit formation process (8) who solves the consumption and portfolio choice problem (18). Assume that the interest rate $r_t$ satisfies (37) and that the economy consists of a (locally) risk-free asset, a risky stock and a zero-coupon bond with time to maturity $T_1$. Let the dynamics

\footnote{We note that this economy emerges as a special case of the economy considered by Brennan and Xia (2002).}
of the risky assets be given by \[36\]. Then the optimal amounts of wealth invested in the risky stock and the bond are given by

\[
\pi^*_1, t = -\frac{1}{\sigma_S} \frac{\partial V}{\partial \log M_t} \frac{1}{V_t} \left( \lambda_{1,t} - \frac{\rho}{\sqrt{1 - \rho^2}} \hat{\lambda}_{2,t} \right) \cdot A_t, \tag{40}
\]

\[
\pi^*_2, t = \frac{\hat{\lambda}_{2,t}}{\sigma_r \sqrt{1 - \rho^2 D_{T_1}}} \cdot \frac{\partial V}{\partial \log M_t} \frac{1}{V_t} \cdot A_t - \frac{1}{D_{T_1}} \cdot \frac{\partial V_t}{\partial r_t}, \tag{41}
\]

with \(V_t = \int_0^{T-t} V_{t,h} \, dh\) representing the market value of the future (approximate) optimal consumption stream \(\{c^*_s\}_{t \leq s \leq T}\) and

\[
\hat{\lambda}_{1,t} = \lambda_1 + \beta \frac{\sigma_r \rho \hat{D}_t \hat{P}_t}{1 + \beta P_t}, \tag{42}
\]

\[
\hat{\lambda}_{2,t} = \lambda_2 + \beta \frac{\sigma_r \sqrt{1 - \rho^2} \hat{D}_t \hat{P}_t}{1 + \beta P_t}, \tag{43}
\]

with \(P_t\) given by \[20\] and \(\hat{D}_t\) defined in Appendix A (see \[81\]).

Figure 6(a) shows the first component of the bond portfolio weight \(\pi^*_2, t / A_t\) as a function of age. We call this component the speculative bond portfolio weight. Two counteracting forces determine how this speculative weight evolves over the individual’s life cycle. On the one hand, the available time to incorporate a speculative shock into future consumption declines with age. As a result, the speculative demand decreases as the individual becomes older. A similar reasoning applies to the stock portfolio weight; see Section 4.5. On the other hand, the older the individual, the more sensitive the individual’s relative consumption choice \(\hat{c}^*_t\) (typically) is to interest rate shocks; see Eqn. \[84\] in Appendix A which shows that \(\hat{\lambda}_{2,t} / \gamma\) models the interest rate sensitivity of \(\hat{c}^*_t\). Note that \(\hat{\lambda}_{2,t}\) becomes more negative as the individual ages. This causes the speculative demand to increase with age. The first effect dominates the second effect in Figure 6(a).

Figure 6(b) shows the second component of the bond portfolio weight \(\pi^*_2, t / A_t\) as a function of age. We call this component the hedging bond portfolio weight. The value of the hedging weight is also the result of two counteracting forces: a substitution effect and a horizon effect. On the one hand, the willingness to substitute consumption over time decreases with age. This causes the hedging bond portfolio weight to increase over the life cycle. On the other hand, the longer the horizon \(h\), the larger the impact of a shock in the interest rate will be on the price of future consumption. This causes the
hedging portfolio weight to decrease over the life cycle. Jointly, these two effects lead to a hump-shaped pattern.

[Place Figure 6 about here]

5 Kreps-Porteus Utility

As shown in Appendix B an individual with habit preferences prefers (unrealistically) high unconditional median growth rates of log consumption (especially at high ages) except when his subjective rate of time preference $\delta$ is excessive\footnote{Indeed, as already pointed out by Deaton [1992], an individual with habit preferences derives utility not only from consumption levels but also from consumption growth.}. This section therefore considers a utility specification that disentangles the elasticity of intertemporal substitution from the individual’s coefficient of relative risk aversion. Under this extended preference model, quite remarkably, median consumption growth can be low or moderate even when the individual’s subjective rate of time preference $\delta$ takes on reasonable values.

5.1 Utility Specification

We specify the individual’s utility process $\{U_t\}_{0 \leq t \leq T}$ in terms of the intertemporal aggregator $f$. More specifically, $\{U_t\}_{0 \leq t \leq T}$ satisfies the following integral equation ($0 \leq t \leq T$):

$$U_t \left( \frac{c_t}{h_t} \right) = \mathbb{E}_t \left[ \int_t^T f \left( \frac{c_s}{h_s}, U_s \right) ds \right]. \quad (44)$$

Here, $\mathbb{E}_t$ denotes the expectation conditional upon information at time $t$. The intertemporal aggregator $f$ is assumed to be given by\footnote{If $\varphi = 0$, then (45) reduces to $f \left( \frac{c_t}{h_t}, U_t \right) = \left( 1 + \zeta U_t \right) \log \{ c_t/h_t \} - \frac{\delta}{\zeta} \log \{ 1 + \zeta U_t \}$.}

$$f \left( \frac{c_t}{h_t}, U_t \right) = \left( 1 + \zeta \right) \left[ \left( \frac{c_t}{h_t} \right)^{\varphi} \left( \frac{U_t}{\varphi} \right)^{\frac{\zeta}{\| \varphi \|}} - \delta U_t \right]. \quad (45)$$

Here, $\zeta > -1$ and $\varphi < \min \{ 1, 1/(1 + \zeta) \}$ are preference parameters. We refer to Eqn. (45) as the Kreps-Porteus aggregator\footnote{If $\zeta = 0$ and the habit level $h_t$ equals unity (i.e., $\alpha = \beta = 0$), then $f \left( \frac{c_t}{h_t}, U_t \right)$ reduces to $f \left( \frac{c_t}{h_t}, U_t \right) = \frac{\varphi}{\varphi^\zeta} - \delta U_t$.}

The individual aims to maximize $U_0 \left( \frac{c}{h} \right) = U \left( \frac{c}{h} \right)$.
(see (44)) with $f\left(\frac{c_h}{r}, U_t\right)$ given by (45) subject to the habit formation process (8) and
the individual’s dynamic budget constraint (7).

5.2 Dynamic Consumption and Portfolio Choice

We can solve the individual’s optimization problem by invoking the approach of
Schroder and Skiadas (1999). The next theorem presents the (approximate) optimal
consumption choice.

**Theorem 5.1.** Consider an individual with utility process (44), intertemporal aggregator
(45) and habit formation process (8) who solves the consumption and portfolio choice
problem (18). Let $h^*_t$ be the individual’s habit level implied by substituting the individual’s
optimal past relative consumption choices $\hat{c}^*_s$ ($s \leq t$) into (15) and let $z$ be a scaling
parameter associated with the static budget constraint in (18). Then the individual’s
(approximate) optimal consumption choice $c^*_t$ is given by

$$c^*_t = h^*_t z \exp\left\{ \int_0^t \left( \psi \hat{r}_s + \frac{1}{2} \frac{\lambda^2}{\gamma} \cdot \delta \right) ds + \frac{\lambda}{\gamma} \int_0^t dW_s \right\},$$  \hspace{1cm} (48)

where $\psi = 1 / (1 - \varphi)$ and $\gamma = 1 - \varphi (1 + \zeta)$.

The scaling parameter $z \geq 0$ is determined such that the individual’s original budget
constraint holds with equality.

From (48) one may verify that the sensitivity $q_h$ and volatility $\Sigma_h$ of future
consumption take the same form as in the base model (see Section 4). In the preference
model of this section, the parameter $\psi$ models the individual’s willingness to substitute
consumption over time. Relative risk aversion is thus decoupled from the elasticity of
intertemporal substitution. Figure [7] illustrates the median consumption path as a
function of age for an individual whose preferences combine SDU with the ratio model
of habit formation. As shown by this figure, the growth rates of the individual’s median
consumption path are substantially lower at high ages compared to the case without
SDU. Indeed, if the elasticity of intertemporal substitution is relatively low (as is the
Eqn. (44) is then equivalent to the additive utility specification

$$U_t \left( \frac{c_t}{r} \right) = E_t \left[ \int_t^T e^{-\delta(s-t)} \frac{1}{\varphi} c_h^s ds \right].$$  \hspace{1cm} (47)
case in Figure 7 where \( \psi \) equals zero), the individual is less willing to substitute current consumption for future consumption in order to avoid large future habit levels.

The general expression for the (approximate) optimal portfolio choice under SDU in an economy with two assets given by (25)–(26) remains the same as in the previous section; see, in particular, (33). However, under SDU, long horizons receive smaller value weights in the computation of the portfolio strategy compared to the case without SDU, as wealth accumulation during retirement is not excessive. As a result, an individual whose preferences combine SDU with habit formation invests less in the risky stock than an individual whose preferences are described by the ratio model of habit formation without SDU; see Figure 8 which shows the reduction in the share of wealth invested in the risky stock as a result of superimposing SDU to our base habit formation model.

6 Accuracy of the Approximation Method

The consumption and portfolio strategies presented in Sections 4 and 5 are exact only in the case when \( \beta = 0 \) and/or \( \alpha = \infty \). In all other cases, the consumption and portfolio strategies are approximate based upon linearizing the individual’s static budget constraint in (16) around the constant relative consumption trajectory \( \{\hat{c}_t\}_{0 \leq t \leq T} = x \) (\( x > 0 \)). This section analyzes the approximation error induced by applying a linearization to the static budget constraint.

We consider an individual whose preferences are represented by (44) with aggregator (45) and habit formation process (8). We determine the genuine optimal consumption choice \( c_t^{\text{opt}} \) and optimal portfolio choice \( \pi_t^{\text{opt}} \) by using the method of backward induction; Appendix C provides details on the numerical solution technique. We evaluate the performance of the approximate optimal consumption choice \( c_t^{*} \) by measuring the relative decline in certainty equivalent consumption.\(^{28}\) Table 2 reports our results. We find that the approximation error is a decreasing function of \( \gamma \), and an

\(^{28}\)The certainty equivalent of an uncertain consumption strategy is defined to be the constant consumption level that yields indifference to the uncertain consumption strategy. The certainty equivalent consumption choice \( ce \) always exists if \( \alpha \geq \beta \). In particular, lifetime utility \( U (c/h) \) is increasing in certainty equivalent consumption \( ce \) if \( \beta \int_0^T e^{-\alpha t} \, dt \leq 1 \). If \( T \) is large, then \( \int_0^T e^{-\alpha t} \, dt \approx \frac{1}{\alpha} \). Hence, we can always compute (for any \( T \)) the certainty equivalent consumption choice \( ce \) if \( \frac{\alpha}{\beta} \leq 1 \).
increasing function of $\beta$. Indeed, if $\gamma$ is large, consumption stays close to the habit level. Also, if $\beta$ is small, habit formation is rather limited. In nearly all cases, the approximation error is smaller than 1%. Finally, we note that Table 2 only considers cases for which habit persistence is maximal (i.e., $\beta/\alpha = 1$). If $\beta$ is smaller than $\alpha$, the welfare loss will be lower. In particular, in the limiting case $\beta = 0$, the welfare loss will vanish. For illustration purposes, Figure 9 also compares, for three different economic scenarios, the optimal consumption path with the approximate consumption path. We observe a close match.

7 Concluding Remarks

This paper has explored how an individual who derives utility from the ratio between his consumption and an endogenous habit should consume and invest over the life cycle. It is well-known that analytical closed-form solutions to internal habit formation models with multiplicative habits do not exist in general. Therefore, we have developed a general solution procedure based on a linearization to the static budget constraint enabling us to transform consumption and portfolio problems with ratio habits into approximate consumption and portfolio problems without habit. We have applied our solution procedure to three important cases of multiplicative habit formation. The first case considers a constant investment opportunity set and assumes that the individual has time-separable preferences in terms of relative consumption and constant relative risk aversion. We have shown that the individual’s preferences induce clearly interpretable implications: the coefficient of relative risk aversion controls the year-on-year volatility of current consumption and the strength of habit persistence controls the extent to which a stock return shock impacts future growth rates of consumption. The second case is an extension that allows for stochastic interest rates and stock-bond investments. We have shown that the speculative bond portfolio weight typically declines with age and that the hedging bond portfolio weight displays a hump-shaped pattern over the life cycle. Finally, we have studied an individual whose preferences combine habit formation with stochastic differential utility. Interestingly, median consumption now no longer grows at unrealistically high rates at high ages and risky assets become less attractive.
A Proofs

A.1 Proof of Theorem 3.1

This appendix discusses how to approximate the left-hand side of the new static budget constraint in (16) around the constant consumption trajectory \( \{ \hat{c}_t \}_{0 \leq t \leq T} = x \) for some positive \( x \). (In the main text we make the simplifying assumption that \( \{ \hat{c}_t \}_{0 \leq t \leq T} = 1 \).) The partial derivative of the new static budget constraint in (16) with respect to the current relative consumption choice \( \hat{c}_t \) is given by

\[
\frac{\partial}{\partial \hat{c}_t} \mathbb{E} \left[ \int_0^T M_t h_t \hat{c}_t \, dt \right] = M_t h_t \, dt + \mathbb{E}_t \left[ \int_t^T M_s \frac{\partial h_s}{\partial \hat{c}_t} \hat{c}_s \, ds \right]. \tag{49}
\]

The partial derivative of the future habit level \( h_s \) \((s \geq t)\) with respect to the current relative consumption choice \( \hat{c}_t \) is given by (this equation follows from differentiating (15) with respect to \( \hat{c}_t \))

\[
\frac{\partial h_s}{\partial \hat{c}_t} = \beta \exp \{ -(\alpha - \beta)(s - t) \} \frac{h_s}{\hat{c}_t} \, dt. \tag{50}
\]

Substituting (50) into (49) and evaluating (49) around the constant consumption trajectory \( \{ \hat{c}_t \}_{0 \leq t \leq T} = x \), we arrive at

\[
\frac{\partial}{\partial \hat{c}_t} \mathbb{E} \left[ \int_0^T M_t h_t \hat{c}_t \, dt \right] = M_t x Q_{t-1} \, dt + \beta \mathbb{E}_t \left[ \int_t^T M_s x Q_{s-1} e^{-(\alpha - \beta)(s-t)} \, ds \right] \, dt \tag{51}
\]

\[
= M_t x Q_{t-1} \left( 1 + \beta P_t \right) \, dt.
\]

Here, we define

\[
Q_t = 1 + \frac{\beta}{\alpha - \beta} \left[ 1 - \exp \{ -(\alpha - \beta)t \} \right], \tag{52}
\]

\[
P_t = \mathbb{E}_t \left[ \int_t^T \frac{M_s x Q_{s-1}}{M_t x Q_{t-1}} e^{-(\alpha - \beta)(s-t)} \, ds \right]. \tag{53}
\]
Hence, we can, by virtue of Taylor series expansion, approximate the left-hand side of the new static budget constraint in (16) by

\[
E \left[ \int_0^T M_t h_t \hat{c}_t \, dt \right] \approx E \left[ \int_0^T M_t x^{Q_t} \, dt \right] \\
+ E \left[ \int_0^T M_t x^{Q_t-1} (1 + \beta P_t) (\hat{c}_t - x) \, dt \right] \\
= -\beta E \left[ \int_0^T M_t x^{Q_t} P_t \, dt \right] \\
+ E \left[ \int_0^T M_t x^{Q_t-1} (1 + \beta P_t) \hat{c}_t \, dt \right].
\]  

(54)

We can now obtain the approximate optimization problem (18) as follows.

1. We replace the left-hand side of the new static budget constraint in (16) by (54).
2. Second, we eliminate the constant term \(-\beta E \left[ \int_0^T M_t x^{Q_t} P_t \, dt \right]\) from (54). This term does not play a role in determining the first-order optimality condition.
3. Finally, we redefine initial wealth \(A_0\) to be \(\hat{A}_0\) such that the approximate optimal consumption strategy \(\{c_t^*\}_{0 \leq t \leq T} = \{h_t^* \hat{c}_t^*\}_{0 \leq t \leq T}\) is budget-feasible. That is,

\[
E \left[ \int_0^T M_t h_t^* \hat{c}_t^* \, dt \right] = A_0.
\]  

(55)

Straightforward computations show that the initial wealth \(\hat{A}_0\) associated with the approximate problem is then given by

\[
\hat{A}_0 = A_0 + E \left[ \int_0^T \hat{M}_t \hat{c}_t^* \, dt \right] - E \left[ \int_0^T M_t h_t^* \hat{c}_t^* \, dt \right].
\]  

(56)

Here, \(\hat{M}_t = M_t (1 + \beta P_t)\). Note that the value of \(\hat{A}_0\) can only be determined after the problem has been solved.
A.2 Proof of Theorem 4.1

Define $\hat{M}_t = M_t x^{Q_t-1} (1 + \beta P_t)$. The Lagrangian $L$ is given by

\begin{equation}
L = E \left[ \int_0^T e^{-\delta t} \frac{1}{1 - \gamma} \hat{C}_t^{1-\gamma} \, dt \right] + y \left( \hat{A}_0 - E \left[ \int_0^T \hat{M}_t \hat{C}_t \, dt \right] \right) \\
= \int_0^T \mathbb{E} \left[ e^{-\delta t} \frac{1}{1 - \gamma} (\hat{C}_t)^{1-\gamma} - y \hat{M}_t \hat{C}_t \right] \, dt + y \hat{A}_0.
\end{equation}

(57)

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize $e^{-\delta t} \frac{1}{1 - \gamma} (\hat{C}_t)^{1-\gamma} - y \hat{M}_t \hat{C}_t$. The approximate optimal relative consumption choice $\hat{C}_t^*$ satisfies the following first-order optimality condition:

\begin{equation}
e^{-\delta t} (\hat{C}_t^*)^{-\gamma} = y \hat{M}_t.
\end{equation}

(58)

After solving the first-order optimality condition, we obtain the following maximum:

\begin{equation}
\hat{C}_t^* = \left( e^{\delta t} y \hat{M}_t \right)^{-\frac{1}{\gamma}}.
\end{equation}

(59)

Hence (use (17)),

\begin{equation}
c_t^* = h_t^* \left( ye^{\delta t} \hat{M}_t \right)^{-\frac{1}{\gamma}}.
\end{equation}

(60)

A standard verification that the optimal solution to the Lagrangian equals the optimal solution to the static problem (see, e.g., Karatzas and Shreve 1998, p. 103) completes the proof.

A.3 Derivation of (27) and (30)

This appendix writes the individual’s consumption choice $c_t^*$ in terms of unexpected past stock return shocks. We can write the stochastic discount factor $\hat{M}_t = M_t x^{Q_t-1} (1 + \beta P_t)$ as follows (this follows from applying Itô’s lemma to $\hat{M}_t = f (M_t, P_t, Q_t) = M_t x^{Q_t-1} (1 + \beta P_t)$):

\begin{equation}
\hat{M}_t = \hat{M}_0 \exp \left\{ - \int_0^t \left( \hat{r}_s + \frac{1}{2} \lambda^2 \right) \, ds \right\} \exp \left\{ -\lambda \int_0^t dW_s \right\},
\end{equation}

(61)

where

\begin{equation}
\hat{r}_s = \beta + \frac{\hat{r}_s - \alpha \beta P_s}{1 + \beta P_s}
\end{equation}

(62)
with \( \tilde{r}_s = r - \beta e^{-(\alpha - \beta)s} \log x \).

Substituting (61) into (24), we arrive at

\[
\hat{c}_t^* = \frac{c_t^*}{h_t^*} = \exp \left\{ \frac{1}{\gamma} \int_0^t \left( \tilde{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds + \frac{\bar{y}}{\gamma} \right\} \exp \left\{ \frac{\lambda}{\gamma} \int_0^t dW_s \right\}.
\]

(63)

Here, \( \bar{y} = - \left( \log y + \log \tilde{M}_0 \right) \).

We can write the habit level \( h_t^* \) as follows:

\[
h_t^* = \exp \left\{ \int_0^t \beta \exp \left\{ -(\alpha - \beta)(t - s) \right\} \log \hat{c}_s^* ds \right\}
= \exp \left\{ \int_0^t \beta \exp \left\{ -(\alpha - \beta)(t - s) \right\} \left[ \frac{1}{\gamma} \int_0^s \left( \tilde{r}_u + \frac{1}{2} \lambda^2 - \delta \right) du + \frac{\bar{y}}{\gamma} + \frac{\lambda}{\gamma} \int_0^s dW_u \right] ds \right\}
= \exp \left\{ \int_0^t \left( \frac{1}{\gamma} Q_{t-s} - \frac{1}{\gamma} \right) \left( \tilde{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds \right\}
\times \exp \left\{ \left( \frac{1}{\gamma} Q_t - \frac{1}{\gamma} \right) \bar{y} + \int_0^t \left( \frac{\lambda}{\gamma} Q_{t-s} - \frac{\lambda}{\gamma} \right) dW_s \right\}.
\]

(64)

Hence,

\[
c_t^* = h_t^* \exp \left\{ \frac{1}{\gamma} \int_0^t \left( \tilde{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds + \frac{\bar{y}}{\gamma} \right\} \exp \left\{ \frac{\lambda}{\gamma} \int_0^t dW_s \right\}
= \exp \left\{ \frac{1}{\gamma} \int_0^t Q_{t-s} \left( \tilde{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds + \frac{1}{\gamma} Q_t \bar{y} + \frac{\lambda}{\gamma} \int_0^t Q_{t-s} dW_s \right\}
= (c_0^*)Q_t \exp \left\{ \frac{1}{\gamma} \int_0^t Q_{t-s} \left( \tilde{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds + \frac{\lambda}{\gamma} \int_0^t Q_{t-s} dW_s \right\}.
\]

(65)

It follows from (65) that

\[
q_{t-s} = \frac{\lambda}{\gamma \sigma} Q_{t-s}
\]

(66)

models the sensitivity of log consumption \( \log c_t^* \) to the unexpected stock return shock \( \sigma dW_s \).

Subtracting \( \log c_{t+h}^* \) from \( \log c_t^* \) and taking the limit \( h \to 0 \), we arrive at (30).
A.4 Proof of Theorem 4.2

Straightforward computations show that

\[ V_{t,h} = \mathbb{E}_t \left[ \frac{M_{t+h}}{M_t} c_{t+h}^* \right] \]

\[ = c_t^* G_{t,h} \mathbb{E}_t \left[ \exp \left\{ - \int_0^h \left( r + \frac{1}{2} \lambda^2 \right) dv - \lambda \int_0^h dW_{t+h-v} \right\} \right] \times \exp \left\{ \frac{1}{\gamma} \int_0^h Q_v \left( \hat{r}_{t+h-v} + \frac{1}{2} \lambda^2 - \delta \right) dv + \frac{\lambda}{\gamma} \int_0^h Q_v dW_{t+h-v} \right\} \]

\[ = c_t^* G_{t,h} C_{t,h}, \]

where

\[ C_{t,h} = \exp \left\{ - \int_0^h \left( r - Q_v \frac{1}{\gamma} \left[ \hat{r}_{t+h-v} + \frac{1}{2} \lambda^2 - \delta \right] + Q_v \frac{\lambda^2}{\gamma} - \frac{1}{2} Q_v^2 \frac{\lambda^2}{\gamma^2} \right) dv \right\}, \]

\[ G_{t,h} = (c_0^*)^{(Q_{t+h} - Q_t)} \exp \left\{ \frac{1}{\gamma} \int_0^t (Q_{t+h-s} - Q_{t-s}) \left( \hat{r}_s + \frac{1}{2} \lambda^2 - \delta \right) ds \right\} \times \exp \left\{ \frac{\lambda}{\gamma} \int_0^t (Q_{t+h-s} - Q_{t-s}) dW_s \right\}. \]

Eqn. (67) shows that the term \( V_{t,h} / c_t^* \) consists of two factors. The factor \( G_{t,h} \) represents past stock return shocks that the individual absorbs into future growth rates of (median) consumption. This factor equals unity if the individual directly absorbs unexpected stock returns shocks into current consumption. The factor \( C_{t,h} \) summarizes the impacts of the unconditional growth rates of median consumption and the future (uncertain) rates of return on the market value of future consumption.

It follows from Itô’s lemma that \( \log V_t = \log \left[ \int_0^{T-t} V_{t,h} \, dh \right] \) satisfies

\[ d \log V_t = (\ldots) \, dt + \frac{\lambda}{\gamma} \int_0^{T-t} Q_h V_{t,h} V_t \, dh \cdot dW_t, \]

suppressing the drift term for brevity. It also holds that (this follows from applying Itô’s lemma to the dynamic budget constraint (7))

\[ d \log A_t = (\ldots) \, dt + \sigma \cdot \frac{\pi_t}{A_t} \cdot dW_t. \]

Setting Eqn. (71) equal to Eqn. (70) and solving for the approximate optimal portfolio choice, we arrive at (33).
A.5 Derivation of (34)

The individual’s approximate optimization problem is given by

\[
\max_{\hat{c}_t, 0 \leq t \leq T_{\text{max}}} \quad \mathbb{E} \left[ \int_0^{T_{\text{max}}} e^{-\delta t} e^{-\int_0^t H_s \, ds} \frac{1}{1 - \gamma} \left( \hat{c}_t \right)^{1-\gamma} \, dt \right]
\]

s.t. \( \mathbb{E} \left[ \int_0^{T_{\text{max}}} \hat{M}_t \hat{c}_t \, dt \right] \leq \hat{A}_0. \) \( (72) \)

Here, \( T_{\text{max}} \) denotes the maximum adult age the individual can reach.

The Lagrangian \( \mathcal{L} \) is given by

\[
\mathcal{L} = \mathbb{E} \left[ \int_0^{T_{\text{max}}} e^{-\delta t} e^{-\int_0^t H_s \, ds} \frac{1}{1 - \gamma} \left( \hat{c}_t \right)^{1-\gamma} \, dt \right] + y \left( \hat{A}_0 - \mathbb{E} \left[ \int_0^{T_{\text{max}}} \hat{M}_t \hat{c}_t \, dt \right] \right)
\]

\( (73) \)

Here, \( y \geq 0 \) denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize \( e^{-\delta t} e^{-\int_0^t H_s \, ds} \frac{1}{1 - \gamma} \left( \hat{c}_t \right)^{1-\gamma} - y \hat{M}_t \hat{c}_t \). The approximate optimal relative consumption choice \( \hat{c}_t^\ast \) satisfies the following first-order optimality condition:

\[
e^{-\delta t} e^{-\int_0^t H_s \, ds} \left( \hat{c}_t^\ast \right)^{-\gamma} = y \hat{M}_t.
\]

(74)

After solving the first-order optimality condition, we obtain the following maximum:

\[
\hat{c}_t^\ast = \left( e^{\delta t} e^{\int_0^t H_s \, ds} y \hat{M}_t \right)^{-\frac{1}{\gamma}}.
\]

(75)

Hence (use (17)),

\[
c_t^\ast = h_t^\ast \left( ye^{\delta t} e^{\int_0^t H_s \, ds} \hat{M}_t \right)^{-\frac{1}{\gamma}}.
\]

(76)

We can now derive the consumption dynamics (34) similarly as in the proof of (30).

A.6 Proof of Theorem 4.3

We first write the individual’s consumption choice \( c_t^\ast \) in terms of unexpected past stock return and interest rate shocks. The stochastic discount factor \( \hat{M}_t = M_t \left( 1 + \beta P_t \right) \)
is given by (this follows from applying Itô’s lemma to $\tilde{M}_t = f(M_t, P_t) = M_t (1 + \beta P_t)$)\footnote{For the sake of simplicity, we assume $x = 1.$} \[ \tilde{M}_t = \tilde{M}_0 \exp \left\{ - \int_0^t \left( \tilde{r}_s + \frac{1}{2} \left\| \tilde{\lambda}_s \right\|^2 \right) ds \right\} \exp \left\{ - \tilde{\lambda}_s^\top \int_0^t dW_s \right\}, \quad (77) \]

where

\[ \tilde{r}_s = \beta + \frac{r_s - \alpha \beta P_s}{1 + \beta P_s}, \quad (78) \]
\[ \tilde{\lambda}_{1,s} = \lambda_1 + \beta \frac{\sigma_r \tilde{D}_s P_s}{1 + \beta P_s}, \quad (79) \]
\[ \tilde{\lambda}_{2,s} = \lambda_2 + \beta \frac{\sigma_r \sqrt{1 - \rho^2} \tilde{D}_s P_s}{1 + \beta P_s}, \quad (80) \]

with

\[ \tilde{D}_s = \int_0^{T-s} \alpha_{s,h} D_h dh. \quad (81) \]

Here,

\[ D_h = \frac{1 - \exp \{ -\kappa h \}}{\kappa}, \quad (82) \]
\[ \alpha_{s,h} = \frac{e^{\int_0^h (\alpha - \beta + r_s + \kappa D_u (\bar{r} - r_s) - \sigma_r D_u (\lambda_1 + \lambda_2 \sqrt{1 - \rho^2}) - \frac{1}{2} \sigma_r^2 D_u^2) du}}{\int_0^{T-s} e^{\int_0^h (\alpha - \beta + r_s + \kappa D_u (\bar{r} - r_s) - \sigma_r D_u (\lambda_1 + \lambda_2 \sqrt{1 - \rho^2}) - \frac{1}{2} \sigma_r^2 D_u^2) du} dh}. \quad (83) \]

Substituting (77) into (24), we arrive at

\[ \tilde{c}_t^\ast = \frac{c_t}{h_t^\ast} = \exp \left\{ \frac{1}{\gamma} \int_0^t \left( \tilde{r}_s + \frac{1}{2} \left\| \tilde{\lambda}_s \right\|^2 - \bar{y} \right) ds + \frac{\bar{y}}{\gamma} \right\} \exp \left\{ \frac{1}{\gamma} \tilde{\lambda}_s^\top \int_0^t dW_s \right\}. \quad (84) \]

Here, \( \bar{y} = - \left( \log y + \log \tilde{M}_0 \right). \)
We express the habit level $h_t^*$ as follows:

$$h_t^* = \exp \left\{ \int_0^t \beta \exp \{-(\alpha - \beta)(t - s)\} \log \hat{c}_s^* ds \right\}$$

$$= \exp \left\{ \int_0^t \beta \exp \{-(\alpha - \beta)(t - s)\} \right\}$$

$$= \exp \left\{ \int_0^t \left( \frac{1}{\gamma} \int_0^s \left( \hat{r}_u + \frac{1}{2} \left\| \hat{\lambda}_u \right\|^2 - \delta \right) du + \frac{\bar{y}}{\gamma} + \frac{1}{\gamma} \hat{\lambda}_u \int_0^s dW_u \right) ds \right\}$$

$$= \exp \left\{ \int_0^t \left( \frac{1}{\gamma} Q_{t-s} - \frac{1}{\gamma} \right) \left( \hat{r}_s + \frac{1}{2} \left\| \hat{\lambda}_s \right\|^2 - \delta \right) ds \right\}$$

$$\times \exp \left\{ \left( \frac{1}{\gamma} Q_t - \frac{1}{\gamma} \right) \bar{y} + \int_0^t \left( \frac{1}{\gamma} Q_{t-s} \hat{\lambda}_s^\top - \frac{1}{\gamma} \hat{\lambda}_s^\top \right) dW_s \right\}. \quad (85)$$

Hence,

$$c_t^* = h_t^* \exp \left\{ \frac{1}{\gamma} \int_0^t \left( \hat{r}_s + \frac{1}{2} \left\| \hat{\lambda}_s \right\|^2 - \delta \right) ds + \frac{\bar{y}}{\gamma} \right\} \exp \left\{ \frac{1}{\gamma} \hat{\lambda}_s^\top \int_0^t dW_s \right\}$$

$$= (c_0^*)^{Q_t} \exp \left\{ \frac{1}{\gamma} \int_0^t Q_{t-s} \left( \hat{r}_s + \frac{1}{2} \left\| \hat{\lambda}_s \right\|^2 - \delta \right) ds + \frac{1}{\gamma} \int_0^t Q_{t-s} \hat{\lambda}_s^\top dW_s \right\}. \quad (86)$$

The market value at time $t$ of the future consumption stream $\{c_s^*\}_{t \leq s \leq T}$, i.e., $V_t = \int_0^{T-t} V_{t,h} dh$, is a function of the state variables $r_t$ and $\log \hat{M}_t$. It now follows from Itô’s lemma that

$$d \log V_t = (\ldots) dt - \left( \frac{\lambda_{1,t}}{\partial \log \hat{M}_t} \frac{1}{V_t} \sigma_r \frac{\partial V_t}{\partial r_t} \right) dW_{1,t}$$

$$- \left( \frac{\lambda_{2,t}}{\partial \log \hat{M}_t} \frac{1}{V_t} - \sigma_r \sqrt{1 - \rho^2} \frac{\partial V_t}{\partial r_t} \right) dW_{2,t}. \quad (87)$$

It also holds that

$$d \log A_t = (\ldots) dt + \left( \frac{\pi_{1,t}}{A_t} \sigma_S - \frac{\pi_{2,t}}{A_t} \sigma_r \rho D_{T_1} \right) dW_{1,t} - \frac{\pi_{2,t}}{A_t} \sigma_r \sqrt{1 - \rho^2} D_{T_1} dW_{2,t}. \quad (88)$$

Setting Eqn. (88) equal to Eqn. (87) and solving for the approximate optimal portfolio choice, we arrive at (40) and (41).
A.7 Proof of Theorem 5.1

Given \( \hat{A}_0 \), the approximate optimal relative consumption choice \( \hat{c}^*_t \) can be obtained from Schroder and Skiadas (1999). Finally, the approximate optimal consumption choice \( c^*_t \) follows as in Eqn. (17).

B Excessive Median Growth Rates of Consumption

We state the following theorem.

Theorem B.1. Suppose that \( r_t \) is constant and let \( \hat{r}_t \) be defined by \((31)\). Then:

1. The value of \( \hat{r}_t \) increases as the preference parameter \( \beta \) increases, given fixed \( \alpha - \beta \).
2. The value of \( \hat{r}_t \) decreases as the terminal time \( T \) increases. In particular, \( \hat{r}_t \to r \) if \( T \to \infty \).

We first prove that the (partial) derivative of \( \hat{r}_t \) with respect to \( \beta \) is positive given fixed \( \alpha - \beta \). Define \( \eta = \alpha - \beta \). Substituting \( \alpha = \eta + \beta \) into \((31)\), we find

\[
\hat{r}_t = \beta + \frac{r - (\eta + \beta) \beta P_t}{1 + \beta P_t}.
\]  

(89)

The (partial) derivative of \( \hat{r}_t \) with respect to \( \beta \) is given by

\[
\frac{\partial \hat{r}_t}{\partial \beta} = 1 + \frac{-(1 + \beta P_t) (\eta + 2\beta) P_t - (r - (\eta + \beta) \beta P_t) P_t}{(1 + \beta P_t)^2}
= 1 + \frac{-\eta P_t - 2\beta P_t - \eta \beta P_t^2 - 2(\beta P_t)^2 - r P_t + \eta \beta P_t^2 + (\beta P_t)^2}{1 + 2\beta P_t + (\beta P_t)^2}
= 1 + \frac{-\eta P_t - 2\beta P_t - (\beta P_t)^2 - r P_t}{1 + 2\beta P_t + (\beta P_t)^2}.
\]  

(90)

Hence,

\[
\frac{\partial \hat{r}_t}{\partial \beta} \geq 0 \iff \frac{-\eta P_t - 2\beta P_t - (\beta P_t)^2 - r P_t}{1 + 2\beta P_t + (\beta P_t)^2} \geq -1
\]

\[
\iff \eta P_t + 2\beta P_t + (\beta P_t)^2 - r P_t \leq 1 + 2\beta P_t + (\beta P_t)^2
\]

\[
\iff (r + \eta) P_t \leq 1
\]

\[
\iff 1 - \exp \{-(r + \eta)(T - t)\} \leq 1.
\]  

(91)

\footnote{In the derivation of Theorem B.1, we assume \( x = 1 \), so that \( \tilde{r}_t = r \) for all \( t \).}

30
Hence, $\partial \hat{r}_t / \partial \beta$ is positive given fixed $\alpha - \beta$.

Finally, we prove that the (partial) derivative of $\hat{r}_t$ with respect to $T$ is negative. The (partial) derivative of $\hat{r}_t$ with respect to $T$ is given by

$$
\frac{\partial \hat{r}_t}{\partial T} = -r (1 + \beta P_t)^{-2} \frac{\partial P_t}{\partial T} - \alpha \beta (1 + \beta P_t)^{-2} \frac{\partial P_t}{\partial T}.
$$

(92)

Using the fact that $\partial P_t / \partial T$ is positive, we find that $\partial \hat{r}_t / \partial T$ is negative. Furthermore, simple algebra yields that $\hat{r}_t = r$ if $T = \infty$. Here, we use the fact that $P_t \to 1/(r + \alpha - \beta)$ as $T \to \infty$.

Theorem B.1 and the decomposition in (30) imply that current consumption has a large impact on future habit levels if the preference parameter $\beta$ is large. Also, the utility gain of an increase in consumption is smaller when the individual is (relatively) young (i.e., small $t$) than when the individual is (relatively) old (i.e., large $t$). As a result, an individual with habit preferences prefers (unrealistically) high unconditional median growth rates of log consumption (especially at high ages) except when his subjective rate of time preference $\delta$ is excessive.

C Numerical Solution Method

To assess the accuracy of our approximation, we also determine the genuine optimal consumption and portfolio policies using numerical backward induction. Because we only explore the case $\alpha = \beta$, we can reduce the number of state variables from two (i.e., wealth level and habit level) to one (i.e., wealth-to-habit ratio). The first step is to specify discrete points in the state space, called grid points. For each grid point, we determine the optimal relative consumption choice and the optimal portfolio choice. To determine the optimal policies, we need to evaluate the utility value for every combination of relative consumption choice and portfolio choice. The utility value is equal to the sum of current utility and the discounted expected continuation value. Once we have computed the utility value for every combination of relative consumption choice and portfolio choice, we select the maximum utility value. We then use this maximum utility value to solve the previous period’s maximization problem. This process is iterated backwards in time until the entire life cycle problem has been solved. In the last period, the optimal relative consumption choice and the maximum utility value are given by $\hat{c}_T^{opt} = A_T / h_T$ and $(\hat{c}_T^{opt})^{1-\gamma} / (1 - \gamma)$, respectively. This gives us the terminal condition for the backward induction procedure. We use Gaussian quadrature to compute expectations. For points
that do not lie on the state space grid, we evaluate the utility level using cubic spline interpolation.

We now introduce the following notation:

- \( S \): total number of simulations;
- \( \Delta t \): time step;
- \( t_n = n\Delta t \) for \( n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor \).

The floor operator \( \left\lfloor \cdot \right\rfloor \) rounds a number downward to its nearest integer.

To compute the welfare loss associated with the approximate consumption strategy, we apply the following steps:

1. We generate \( S \) trajectories of the stochastic discount factor (\( s = 1, \ldots, S \)):
   \[
   M_{s,t_{n+1}} = M_{s,t_n} - rM_{s,t_n} \Delta t - \lambda M_{s,t_n} \sqrt{\Delta t} \epsilon_{s,t_{n}}, \quad n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor .
   \] (93)

   Here, \( \epsilon_{s,t_{n}} \) is a standard normally distributed random variable.

2. We compute the approximate relative consumption choice \( \hat{c}^*_{s,t_{n}} \) and the approximate portfolio strategy \( \pi^*_{s,t_{n}} \) for \( s = 1, \ldots, S \) and \( n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor \). We note that the approximate relative consumption choice \( \hat{c}^*_{s,t_{n}} \) is a function of the stochastic discount factor \( \hat{M}_{s,t_{n}} = M_{s,t_{n}} (1 + \beta P_{t_{n}}) \). The individual’s lifetime utility \( U(c/h) \) can now be obtained by using the method of numerical backward induction. Note that in this step we use backward induction only to obtain lifetime utility and not to determine the optimal solutions.

3. We numerically solve for the certainty equivalent consumption \( c^* \).

4. We compute the optimal consumption strategy \( c^\text{opt}_{s,t_{n}} \) and the optimal portfolio strategy \( \pi^\text{opt}_{s,t_{n}} \) for \( s = 1, \ldots, S \) and \( n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor \). Lifetime utility follows from the backward induction algorithm.

5. We numerically solve for the optimal certainty equivalent consumption \( c^\text{opt} \).

6. Finally, we compute the welfare loss \( l \):
   \[
   l = \frac{c^\text{opt} - c^*}{c^\text{opt}}.
   \] (94)
References


Figures and Tables

Figure 1: **Sensitivity and annualized volatility of future consumption.** Panel A illustrates the sensitivity of future consumption to a stock return shock (i.e., $q_h$) as a function of the horizon $h$ (i.e., the time distance between the date at which the stock return shock occurs and the date of consumption), while panel B shows the annualized volatility of future consumption (i.e., $\Sigma_h$) as a function of the horizon $h$ (i.e., the time distance between the current time and the date of consumption). The two panels consider four different types of individuals: a highly risk averse individual with a low degree of habit persistence (i.e., $\gamma = 20, \alpha = 0.2, \beta = 0.1$); a highly risk averse individual with a high degree of habit persistence (i.e., $\gamma = 20, \alpha = 0.3, \beta = 0.3$); a moderately risk averse individual with a low degree of habit persistence (i.e., $\gamma = 10, \alpha = 0.2, \beta = 0.1$); and a moderately risk averse individual with a high degree of habit persistence (i.e., $\gamma = 10, \alpha = 0.3, \beta = 0.3$). In the case of a CRRA individual, the sensitivity and the annualized volatility of future consumption do not depend on the horizon $h$. We set both the market price of risk $\lambda$ and the stock return volatility $\sigma$ equal to 0.2.
−40% stock return shock in year 1

+20% stock return shock in year 2 (next to the −40% stock return shock in year 1)

Figure 2: Shock absorbing mechanisms. The figure shows the impact of unexpected stock return shocks on current and future consumption choices. The left panels consider a CRRA individual, while the right panels consider an individual whose preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). The CRRA individual invests 50% of his accumulated wealth in the stock market (i.e., his relative risk aversion coefficient is equal to 2). Wealth at the age of 25 is for both individuals equal to 45. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%.
Figure 3: Consumption dynamics. Panel A illustrates a consumption path of a CRRA individual, while panel B shows a consumption path of an individual whose preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). The CRRA individual invests 50% of his accumulated wealth in the stock market (i.e., his relative risk aversion coefficient is equal to 2). Wealth at the age of 25 is for both individuals equal to 45. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.

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Table 1: Median year-on-year volatility of wealth. The table reports the median year-on-year volatility of wealth for various ages. The year-on-year consumption volatility is always equal to 2%, irrespective of the individual’s age. The individual’s preference parameters are: $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$. Wealth at the age of 25 is equal to 45. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
Figure 4: Investment strategy. The figure shows the investment strategy of an individual whose preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). Wealth at the age of 25 is equal to 45. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
Figure 5: **Growth rate of median consumption.** The figure illustrates the growth rate of median consumption as a function of age for the case where the terminal time $T$ is equal to the individual’s uncertain date of death. The individual preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). Survival rates are taken from the Human Mortality Database. We use the unisex mortality table for the U.S. population for 2015. Wealth at the age of 25 is equal to 45. For comparison purposes, we also plot the growth rate of median consumption for the case with a fixed date of death; see the dotted line. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
Figure 6: **Portfolio choice with stochastic interest rates.** Panel A illustrates the median speculative bond portfolio weight (with and without habit formation) as a function of age. We assume that the individual invests wealth in a zero-coupon bond with a fixed time to maturity of 10 (i.e., $T_1 = 10$). Panel B illustrates the median hedging bond portfolio weight (with and without habit formation) as a function of age. The individual’s preference parameters are as follows: $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$ (for the case of no habit formation, we have $\alpha = \beta = 0$). We set the long-term interest rate $\bar{r}$ equal to 1%, the mean reversion parameter $\kappa$ to 0.1, the interest rate volatility $\sigma_r$ to 2%, the market price of interest rate risk $\lambda_2$ to -0.2, the market price of stock market risk $\lambda_1$ to 0.2, the stock return volatility $\sigma_S$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year. Note that part of the individual’s wealth is invested in the money market account.
Figure 7: Median consumption path. The figure illustrates the median consumption path as a function of age for an individual whose preferences combine SDU with the ratio model of habit formation. The preference parameters are: $\psi = 0$, $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. Wealth at the age of 25 is equal to 45. For comparison purposes, we also plot the median consumption path for the case without SDU; see the dotted line. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
Figure 8: **Reduction in risky stock portfolio weight.** The figure illustrates the reduction in the share of wealth invested in the risky stock (in %) as a result of superimposing SDU to our base habit formation model, as a function of age. The preference parameters are: $\psi = 0$, $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. Wealth at the age of 25 is equal to 45. We set the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
Figure 9: Consumption trajectories. The figure compares, for three different economic scenarios, the optimal consumption path with the approximate consumption path. The preference parameters are: $\psi = 1/10$, $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. Initial wealth equals 15. We set the terminal time $T$ equal to 20, the risk-free interest rate $r$ equal to 1%, the market price of risk $\lambda$ to 0.2, the stock return volatility $\sigma$ to 20%, and the subjective rate of time preference $\delta$ to 3%. The individual adjusts consumption once a year.
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(d) Sensitivity with respect to the Initial Wealth Level $A_0$

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(e) Sensitivity with respect to the Preference Parameter $\psi$

Table 2: Welfare losses. The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the approximate optimal consumption choice (24). We set the terminal time $T$ equal to 20, the risk-free interest rate $r$ to 1%, the market price of risk $\lambda$ to 0.2, and the stock return volatility $\sigma$ to 20%. The individual adjusts consumption once a year.